Linear Algebra - Part 52

We know for $A \in \mathbb{R}^{2\times 2}$: $\det(A) \neq 0 \iff Ax = b$ has a unique solution

$\iff A$ invertible $=$ non-singular

For $A \in \mathbb{R}^{n\times n}$: $\det(A) = 0 \iff A$ singular

Proposition: For $A \in \mathbb{R}^{n\times n}$, the following claims are equivalent:

- $\det(A) \neq 0$
- columns of A are linearly independent
- rows of A are linearly independent
- $\operatorname{rank}(A) = n$
- $\operatorname{Ker}(A) = \{0\}$
- A is invertible
- $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$

Cramer's rule: $A \in \mathbb{R}^{n\times n}$ non-singular, $b \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ unique solution of $Ax = b$.

Then:

$$x_i = \frac{\det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & b & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}}{\det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & a_i & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}}$$

Proof: Use cofactor matrix $C \in \mathbb{R}^{n \times n}$ defined: $c_{ij} = (-1)^{i+j} \cdot \det(A)$ jth column deleted ith row deleted

$$\overset{\text{Laplace expansion}}{=} \det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{j-1} & e_i & a_{j+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}$$

We can show: $$A^{-1} = \frac{C^T}{\det(A)}$$

Hence: $$x = A^{-1} b = \frac{C^T b}{\det(A)}$$ and $$(C^T b)_i = \sum_{k=1}^{n} (C^T)_{ik} b_k = \sum_{k=1}^{n} c_{ki} b_k$$

$$= \sum_{k=1}^{n} \det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & e_k & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix} b_k$$

$$\overset{\text{linear in the } i\text{th column}}{=} \det\begin{pmatrix} | & & | & b_1 & | & & | \\ a_1 & \cdots & a_{i-1} & b_2 & a_{i+1} & \cdots & a_n \\ | & & | & \vdots & | & & | \\ & & & b_n & & & \end{pmatrix}$$

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