Linear Algebra – Part 50

determinant is multiplicative: $\det(MA) = \det(M) \cdot \det(A)$

Gaussian elimination: $A \rightsquigarrow MA$ (row operations; MA ← (see part 37))

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} \begin{pmatrix} — & \alpha_1^T & — \\ — & \alpha_2^T & — \\ — & \alpha_3^T & — \end{pmatrix} = \begin{pmatrix} — & \alpha_1^T & — \\ — & \alpha_2^T & — \\ & \alpha_3^T + \lambda \cdot \alpha_1^T & \end{pmatrix}$$

$$Z_{3+\lambda 1} \Rightarrow \det(Z_{3+\lambda 1}) = 1$$

Adding rows with $Z_{i+\lambda j}$ $(i \neq j,\ \lambda \in \mathbb{R})$ does not change the determinant!

Exchanging rows with $P_{i \leftrightarrow j}$ $(i \neq j)$ does change the sign of the determinant!

Scaling one row with factor d_j scales the determinant by d_j!

Column operations? $\det(A^T) = \det(A)$ ✓

Example:

$$\det \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 4 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} \overset{\text{rows } \mathrm{II} - 1 \cdot \mathrm{V}}{=} \det \begin{pmatrix} +-1 & 1 & +0 & -2 & 0 \\ 0 & 4 & -0 & -2 & 2 \\ 1 & 0 & +0 & -3 & 1 \\ 1 & 2 & -0 & 0 & 3 \\ 0 & -2 & +1 & 1 & 2 \end{pmatrix}$$

Laplace expansion

$$= (+1) \cdot \det \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

columns $\mathrm{II} - 2\,\mathrm{IV}$, $\mathrm{III} + \mathrm{IV}$

$$= \det \begin{pmatrix} -1^+ & 1 & -2 & 0 \\ 0^- & 0^+ & 0^- & 2^+ \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix}$$

Laplace expansion

$$= (+2) \cdot \det \begin{pmatrix} -1 & 1 & -2 \\ 1 & -2 & -2 \\ 1 & -4 & 3 \end{pmatrix} = 2 \cdot 13 = \underline{26}$$