The Bright Side of Mathematics

# Linear Algebra – Part 50

determinant is multiplicative: $\det(MA) = \det(M) \cdot \det(A)$

Gaussian elimination: $A \xrightarrow{\text{row operations}} MA$ (see part 37)

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ — \alpha_3^T — \end{pmatrix} = \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ \alpha_3^T + \lambda \cdot \alpha_1^T \end{pmatrix}$$

$$Z_{3+\lambda 1} \Rightarrow \det(Z_{3+\lambda 1}) = 1$$

Adding rows with $Z_{i+\lambda j}$ $(i \neq j,\ \lambda \in \mathbb{R})$ does not change the determinant!

Exchanging rows with $P_{i \leftrightarrow j}$ $(i \neq j)$ does change the sign of the determinant!

Scaling one row with factor $d_j$ scales the determinant by $d_j$!

Column operations? $\det(A^T) = \det(A)$ ✓

Example:

$$\det \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 4 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} \overset{\substack{\text{rows} \\ \mathrm{II} - 1 \cdot \mathrm{V}}}{=} \det \begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix}$$

$$\overset{\text{Laplace expansion}}{=} (+1) \cdot \det \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

$$\overset{\substack{\text{columns} \\ \mathrm{II} - 2\,\mathrm{IV} \\ \mathrm{III} + \mathrm{IV}}}{=} \det \begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix}$$

$$\overset{\text{Laplace expansion}}{=} (+2) \cdot \det \begin{pmatrix} -1 & 1 & -2 \\ 1 & -2 & -2 \\ 1 & -4 & 3 \end{pmatrix} = 2 \cdot 13 = 26$$