Linear Algebra – Part 48

4x4-matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11}\cdot\det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

24 permutations

$$- a_{21}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

$$+ a_{31}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

$$- a_{41}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

checkerboard

$$\begin{pmatrix} + & - & + & \\ - & + & - & \\ + & - & & \ddots \\ - & + & & \end{pmatrix}$$

Idea: $n\times n \rightsquigarrow (n-1)\times(n-1) \rightsquigarrow \cdots \rightsquigarrow 3\times 3 \rightsquigarrow 2\times 2 \rightsquigarrow 1\times 1$

Laplace expansion: $A \in \mathbb{R}^{n\times n}$. For jth column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij}\cdot \det\left(A^{(i,j)}\right)$$ expanding along the jth column

For ith row:

ith row and jth column are deleted

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\cdot \det\left(A^{(i,j)}\right)$$ expanding along the ith row

Example:

expaning along 2nd row

$$\det\begin{pmatrix} {}^{+}0 & 2 & 3 & 4 \\ {}^{-}2 & {}^{+}0 & {}^{-}0 & {}^{+}0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix} = -2\cdot\det\begin{pmatrix} {}^{+}2 & 3 & 4 \\ {}^{-}1 & {}^{+}0 & {}^{-}0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= (-2)(-1)\cdot 1\cdot\det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2\cdot(6-4) = 4$$