



# Linear Algebra - Part 48

4x4-matrix:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} \quad \text{6 permutations}$$

24 permutations

checkerboard

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & \dots \\ - & + & \dots \end{pmatrix}$$

$$- a_{21} \cdot \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad \text{6 permutations}$$

$$+ a_{31} \cdot \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad \text{6 permutations}$$

$$- a_{41} \cdot \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \quad \text{6 permutations}$$

Idea:  $n \times n \rightsquigarrow (n-1) \times (n-1) \rightsquigarrow \dots \rightsquigarrow 3 \times 3 \rightsquigarrow 2 \times 2 \rightsquigarrow 1 \times 1$

Laplace expansion:  $A \in \mathbb{R}^{n \times n}$ . For  $j$ th column:

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \cdot \det(A^{(i,j)}) \quad \text{expanding along the } j\text{th column}$$

For  $i$ th row:

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \cdot \det(A^{(i,j)}) \quad \text{expanding along the } i\text{th row}$$

Example:

$$\det \begin{pmatrix} +0 & 2 & 3 & 4 \\ -2 & +0 & -0 & +0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix} \stackrel{\text{expanding along 2nd row}}{=} -2 \cdot \det \begin{pmatrix} +2 & 3 & 4 \\ -1 & +0 & -0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= (-2) \cdot (-1) \cdot 1 \cdot \det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2 \cdot (6-4) = 4$$