Linear Algebra – Part 48

4x4-matrix:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

24 permutations

$$- a_{21} \cdot \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

checkerboard

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & \ddots \\ - & + & \end{pmatrix}$$

$$+ a_{31} \cdot \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

$$- a_{41} \cdot \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ 6 permutations

Idea: $n \times n \rightsquigarrow (n-1) \times (n-1) \rightsquigarrow \cdots \rightsquigarrow 3 \times 3 \rightsquigarrow 2 \times 2 \rightsquigarrow 1 \times 1$

Laplace expansion: $A \in \mathbb{R}^{n \times n}$.   For jth column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det\left(A^{(i,j)}\right)$$ expanding along the jth column

For ith row:

ith row and jth column are deleted

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det\left(A^{(i,j)}\right)$$ expanding along the ith row

Example:

expaning along 2nd row

$$\det \begin{pmatrix} \overset{+}{0} & 2 & 3 & 4 \\ \overset{-}{2} & \overset{+}{0} & \overset{-}{0} & 0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix} = -2 \cdot \det \begin{pmatrix} \overset{+}{2} & 3 & 4 \\ \overset{+}{1} & \overset{+}{0} & \overset{-}{0} \\ 0 & 1 & 2 \end{pmatrix}$$

$$= (-2)(-1) \cdot 1 \cdot \det \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2 \cdot (6-4) = 4$$