



## Linear Algebra - Part 46

n-dimensional volume form:  $\text{vol}_n: \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n \text{ times}} \rightarrow \mathbb{R}$

- linear in each entry
- antisymmetric
- $\text{vol}_n(e_1, e_2, \dots, e_n) = 1$

Let's calculate:

$$\text{vol}_n \left( \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right) = \text{vol}_n \left( a_{11} \cdot e_1 + \dots + a_{n1} e_n, \dots \right) \quad (*)$$

$$= a_{11} \cdot \text{vol}_n(e_1, \dots) + \dots + a_{n1} \cdot \text{vol}_n(e_n, \dots)$$

$$= \sum_{j_1=1}^n a_{j_1,1} \text{vol}_n(e_{j_1}, \dots) = \sum_{j_1=1}^n a_{j_1,1} \text{vol}_n \left( e_{j_1}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

$$= \sum_{j_1=1}^n \sum_{j_2=1}^n a_{j_1,1} a_{j_2,2} \cdot \text{vol}_n \left( e_{j_1}, e_{j_2}, \begin{pmatrix} a_{13} \\ \vdots \\ a_{n3} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

$$= \sum_{j_1=1}^n \sum_{j_2=1}^n \dots \sum_{j_n=1}^n a_{j_1,1} a_{j_2,2} \dots a_{j_n,n} \cdot \underbrace{\text{vol}_n(e_{j_1}, e_{j_2}, \dots, e_{j_n})}_{=0 \text{ if two indices coincide}}$$

permutation of  $\{1, \dots, n\}$

$$= \sum_{(j_1, \dots, j_n) \in \mathcal{S}_n} a_{j_1,1} a_{j_2,2} \dots a_{j_n,n} \cdot \underbrace{\text{vol}_n(e_{j_1}, e_{j_2}, \dots, e_{j_n})}_{= \begin{cases} 1 \\ -1 \end{cases}}$$

$(j_1, \dots, j_n) \in \mathcal{S}_n$   
where all entries  
are different

set of all permutations of  $\{1, \dots, n\}$

$$\text{sgn}(j_1, \dots, j_n) = \begin{cases} +1, & \text{even number of exchanges} \\ & \text{to get to } (1, \dots, n) \\ -1, & \text{odd number of exchanges} \\ & \text{to get to } (1, \dots, n) \end{cases}$$

$$= \sum_{(j_1, \dots, j_n) \in \mathcal{S}_n} \text{sgn}(j_1, \dots, j_n) a_{j_1,1} a_{j_2,2} \dots a_{j_n,n} = \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

(Leibniz formula)