Linear Algebra – Part 46

n-dimensional volume form: $\mathrm{vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$

- linear in each entry
- antisymmetric
- $\mathrm{vol}_n(e_1, e_2, \ldots, e_n) = 1$

Let's calculate:

$$\mathrm{vol}_n\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \underbrace{\begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}}_{(*)} \right) = \mathrm{vol}_n\left(a_{11} \cdot e_1 + \cdots + a_{n1} e_n , (*) \right)$$

$$= a_{11} \cdot \mathrm{vol}_n(e_1, (*)) + \cdots + a_{n1} \cdot \mathrm{vol}_n(e_n, (*))$$

$$= \sum_{j_1 = 1}^{n} a_{j_1,1} \, \mathrm{vol}_n(e_{j_1}, (*)) = \sum_{j_1 = 1}^{n} a_{j_1,1} \, \mathrm{vol}_n\left(e_{j_1}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

$$= \sum_{j_1 = 1}^{n} \sum_{j_2 = 1}^{n} a_{j_1,1} a_{j_2,2} \cdot \mathrm{vol}_n\left(e_{j_1}, e_{j_2}, \begin{pmatrix} a_{13} \\ \vdots \\ a_{n3} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

$$= \sum_{j_1 = 1}^{n} \sum_{j_2 = 1}^{n} \cdots \sum_{j_n = 1}^{n} a_{j_1,1} a_{j_2,2} \cdots a_{j_n,n} \cdot \underbrace{\mathrm{vol}_n(e_{j_1}, e_{j_2}, \ldots, e_{j_n})}_{= 0 \text{ if two indices coincide}}$$

$$= \sum_{(j_1, \ldots, j_n) \in S_n} a_{j_1,1} a_{j_2,2} \cdots a_{j_n,n} \cdot \underbrace{\mathrm{vol}_n(e_{j_1}, e_{j_2}, \ldots, e_{j_n})}_{= \begin{cases} 1 \\ -1 \end{cases}}$$

permutation of $\{1, \ldots, n\}$ → $(j_1, \ldots, j_n) \in S_n$: where all entries are different

S_n: set of all permutations of $\{1, \ldots, n\}$

$$\mathrm{sgn}((j_1, \ldots, j_n)) = \begin{cases} +1 , & \text{even number of exchanges to get to } (1, \ldots, n) \\ -1 , & \text{odd number of exchanges to get to } (1, \ldots, n) \end{cases}$$

$$= \sum_{(j_1, \ldots, j_n) \in S_n} \mathrm{sgn}((j_1, \ldots, j_n)) \, a_{j_1,1} a_{j_2,2} \cdots a_{j_n,n} = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

(Leibniz formula)