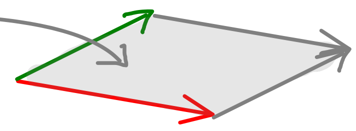
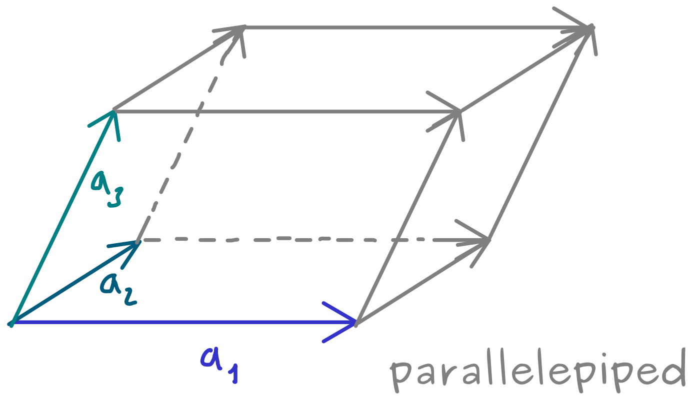
Linear Algebra - Part 45

volume measure? • area in $\mathbb{R}^2$

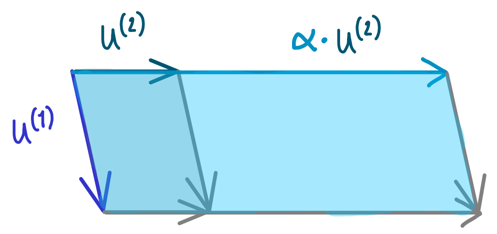


• n-dimensional volume $\mathbb{R}^n$



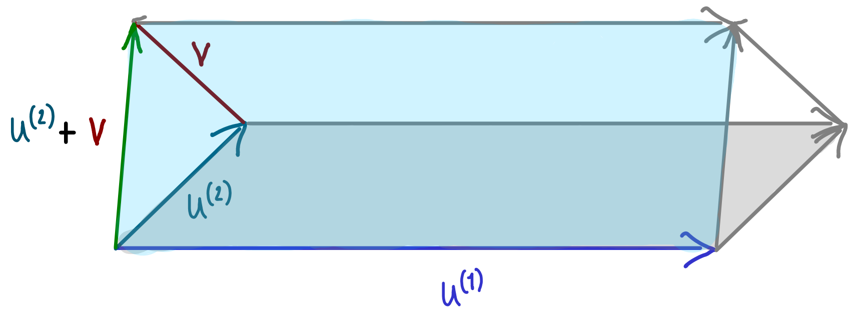
Definition: $\mathrm{vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$ is called n-dimensional volume function if

(a) $\mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, \alpha \cdot u^{(j)}, \ldots, u^{(n)}\big) = \alpha \cdot \mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(n)}\big)$



for all $u^{(1)}, \ldots, u^{(n)} \in \mathbb{R}^n$
for all $\alpha \in \mathbb{R}$
for all $j \in \{1, \ldots, n\}$

(b) $\mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, u^{(j)} + v, \ldots, u^{(n)}\big) = \mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(n)}\big)$
$+ \mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, v, \ldots, u^{(n)}\big)$



for all $u^{(1)}, \ldots, u^{(n)} \in \mathbb{R}^n$
for all $v \in \mathbb{R}^n$
for all $j \in \{1, \ldots, n\}$

(c) $\mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, u^{(i)}, \ldots, u^{(j)}, \ldots, u^{(n)}\big)$

$= - \mathrm{vol}_n\big(u^{(1)}, u^{(2)}, \ldots, u^{(j)}, \ldots, u^{(i)}, \ldots, u^{(n)}\big)$ for all $u^{(1)}, \ldots, u^{(n)} \in \mathbb{R}^n$
for all $i, j \in \{1, \ldots, n\}$
$i \neq j$

(d) $\mathrm{vol}_n(e_1, e_2, \ldots, e_n) = 1$ (unit hypercube)

Result in $\mathbb{R}^2$:

$$\mathrm{vol}_2\left(\begin{pmatrix}a\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) = \mathrm{vol}_2\left(\begin{pmatrix}a\\0\end{pmatrix}+\begin{pmatrix}0\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right)$$

$$\overset{(b)}{=} \mathrm{vol}_2\left(\begin{pmatrix}a\\0\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) + \mathrm{vol}_2\left(\begin{pmatrix}0\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right)$$

$$\overset{(a)}{=} a\cdot\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) + c\cdot\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right)$$

$$\overset{(b)}{=} a\cdot\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}b\\0\end{pmatrix}\right) + a\cdot\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\d\end{pmatrix}\right) + c\cdot\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}b\\0\end{pmatrix}\right) + c\cdot\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\d\end{pmatrix}\right)$$

$$\overset{(b)}{=} a\cdot b\,\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)}_{=0} + a\cdot d\,\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)}_{\overset{(d)}{=}1} + c\cdot b\,\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)}_{=-1} + c\cdot d\cdot\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)}_{=0}$$

$$\overset{(c),(d)}{=} a\cdot d - b\cdot c = \det\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

Define: $$\det\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} = \mathrm{vol}_n\left(\begin{pmatrix}a_{11}\\ \vdots\\ a_{n1}\end{pmatrix},\begin{pmatrix}a_{12}\\ \vdots\\ a_{n2}\end{pmatrix},\ldots,\begin{pmatrix}a_{1n}\\ \vdots\\ a_{nn}\end{pmatrix}\right)$$