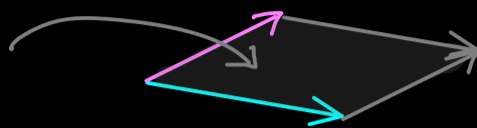


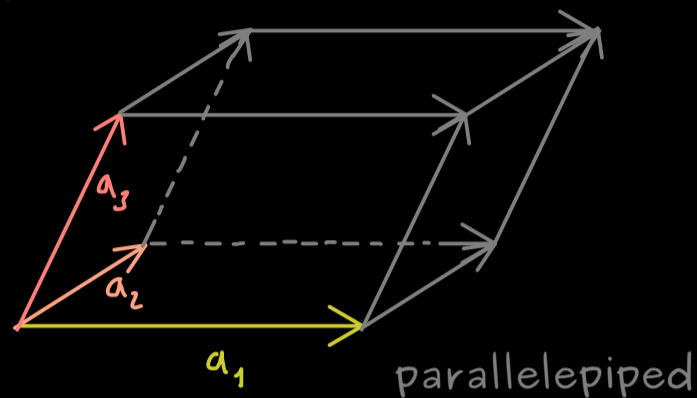
Linear Algebra - Part 45

volume measure?

• area in \mathbb{R}^2



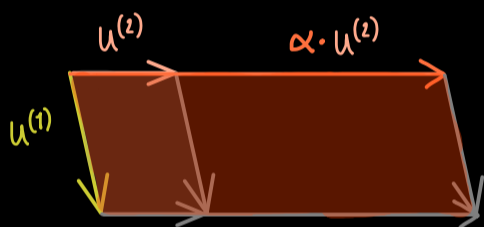
• n-dimensional volume \mathbb{R}^n



parallelepiped

Definition: $\text{vol}_n: \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$ is called n-dimensional volume function if

$$(a) \text{vol}_n(u^{(1)}, u^{(2)}, \dots, \alpha \cdot u^{(j)}, \dots, u^{(n)}) = \alpha \cdot \text{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(n)})$$



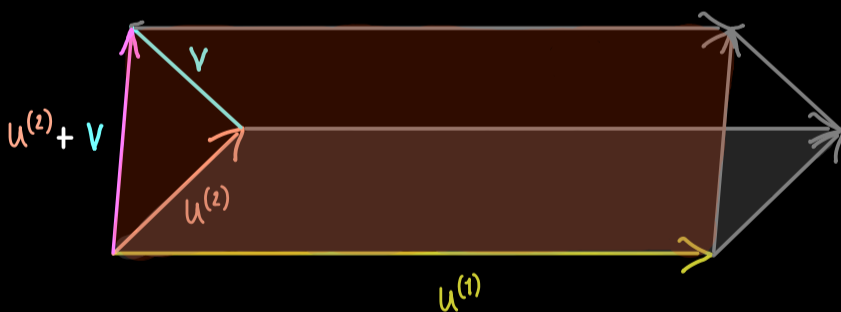
for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$

for all $\alpha \in \mathbb{R}$

for all $j \in \{1, \dots, n\}$

$$(b) \text{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)} + v, \dots, u^{(n)}) = \text{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(n)})$$

$$+ \text{vol}_n(u^{(1)}, u^{(2)}, \dots, v, \dots, u^{(n)})$$



for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$

for all $v \in \mathbb{R}^n$

for all $j \in \{1, \dots, n\}$

$$(c) \text{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(i)}, \dots, u^{(j)}, \dots, u^{(n)})$$

$$= - \text{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(i)}, \dots, u^{(n)})$$

for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$

for all $i, j \in \{1, \dots, n\}$

$i \neq j$

$$(d) \text{vol}_n(e_1, e_2, \dots, e_n) = 1 \quad (\text{unit hypercube})$$

Result in \mathbb{R}^2 :

$$\text{vol}_2 \left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right) = \text{vol}_2 \left(\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right)$$

$$\stackrel{(b)}{=} \text{vol}_2 \left(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right) + \text{vol}_2 \left(\begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right)$$

$$\stackrel{(a)}{=} a \cdot \text{vol}_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right) + c \cdot \text{vol}_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right)$$

$$\stackrel{(b)}{=} a \cdot \text{vol}_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix} \right) + a \cdot \text{vol}_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix} \right) + c \cdot \text{vol}_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix} \right) + c \cdot \text{vol}_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix} \right)$$

$$\stackrel{(b)}{=} a \cdot b \underbrace{\text{vol}_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)}_{=0} + a \cdot d \underbrace{\text{vol}_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)}_{\stackrel{(d)}{=} 1} + c \cdot b \underbrace{\text{vol}_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)}_{=-1} + c \cdot d \cdot \underbrace{\text{vol}_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)}_{=0}$$

$$\stackrel{(c),(d)}{=} a \cdot d - b \cdot c = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{Define: } \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \text{vol}_n \left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$