Linear Algebra – Part 44

$A \in \mathbb{R}^{2\times 2} \rightsquigarrow$ system of linear equations $Ax = b$

Assume $a_{11} \neq 0$

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) \xrightarrow{\mathrm{II} - \frac{a_{21}}{a_{11}} \mathrm{I}} \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array}\right) \xrightarrow{\mathrm{II} \cdot a_{11}}$$

$$\rightsquigarrow \left(\begin{array}{cc|c} \boxed{a_{11}} & a_{12} & b_1 \\ 0 & \boxed{a_{11}a_{22} - a_{21}a_{12}} & a_{11} \cdot b_2 - a_{21} b_1 \end{array}\right)$$

$\neq 0 \iff$ we have a unique solution

<u>Definition:</u> For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, the number

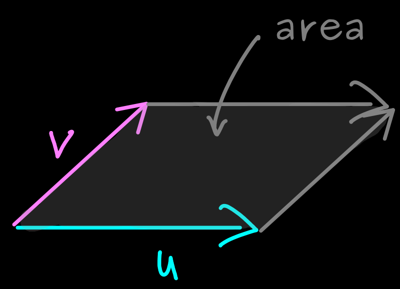
$$\det(A) := a_{11} a_{22} - a_{12} a_{21}$$

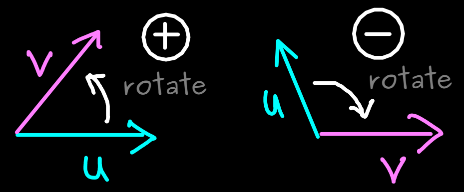
is called the <u>determinant of A</u>.

<u>What about volumes?</u> $\rightsquigarrow$ vol_n

in $\mathbb{R}^2$: $\mathrm{vol}_2(u, v) :=$ <u>orientated</u> area of parallelogram

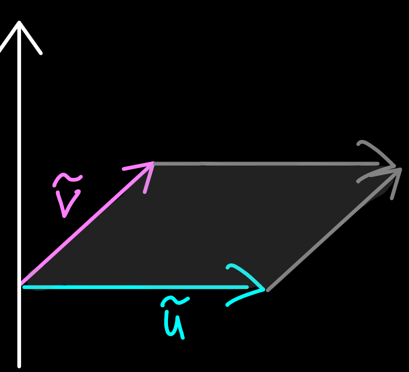
$\pm$





<u>Relation to cross product:</u> embed $\mathbb{R}^2$ into $\mathbb{R}^3$: $\tilde{u} := \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$, $\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

$\mathbb{R}^3$:



$$\| \tilde{u} \times \tilde{v} \| = \left\| \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \right\| = \underbrace{| u_1 v_2 - v_1 u_2 |}_{\det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}}$$

<u>Result:</u> $\mathrm{vol}_2(u, v) = \det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$ (volume function = determinant)