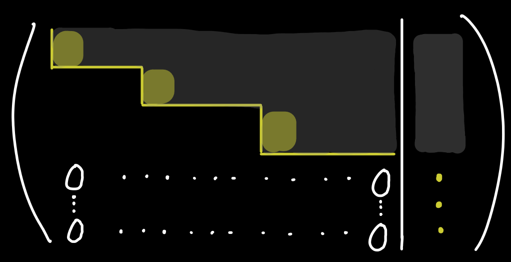
Linear Algebra – Part 42

$Ax = b \rightsquigarrow$ row echelon form



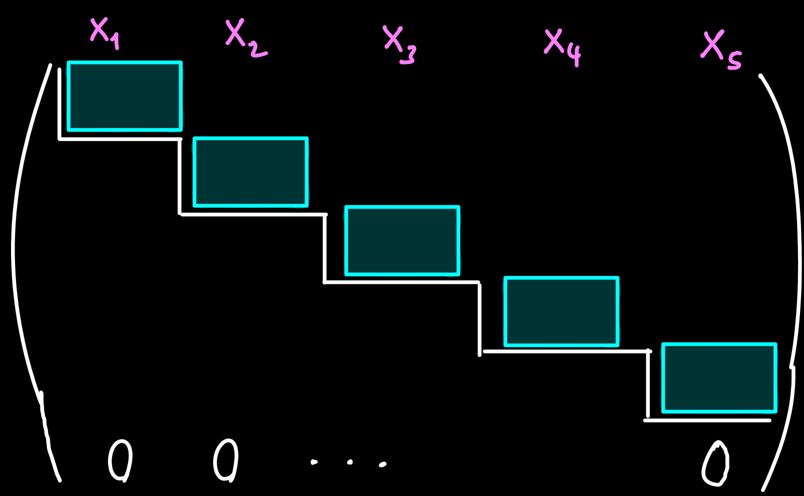
$$S = \emptyset \quad \text{or} \quad S = v_0 + \mathrm{Ker}(A)$$

Proposition: For $A \in \mathbb{R}^{m \times n}$, we have the following equivalences:

(a) For every $b \in \mathbb{R}^m$: $Ax = b$ has at most one solution.

(b) $\mathrm{Ker}(A) = \{0\}$

(c) Row echelon form looks like:

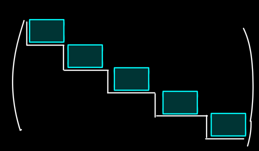


$\boxed{\text{every column has a pivot}}$

(d) $\mathrm{rank}(A) = n$

(e) The linear map $f_A: \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ is injective.

Result for square matrices: For $A \in \mathbb{R}^{n \times n}$:



$$\begin{array}{ccccc}
\mathrm{Ker}(A) = \{0\} & \Longleftrightarrow & \mathrm{Ran}(A) = \mathbb{R}^n & \Longleftrightarrow & Ax = b \text{ has a unique solution for some } b \in \mathbb{R}^n \\
\Updownarrow & & \Updownarrow & & \\
f_A \text{ injective} & \Longleftrightarrow & f_A \text{ surjective} & \Longleftrightarrow & Ax = b \text{ has a unique solution for all } b \in \mathbb{R}^n
\end{array}$$