Linear Algebra - Part 40

Row echelon form

$$A = \begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & \boxed{4} & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: A matrix $A \in \mathbb{R}^{m \times n}$ is in row echelon form if:

(1) All zero rows (if there are any) are at the bottom.

(2) For each row: the first non-zero entry is strictly to the right of the first non-zero entry of the row above.

pivots

$$A = \begin{pmatrix} \boxed{1} & 3 & 5 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition:

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 5 & 0 & 1 \\ 0 & \boxed{2} & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

variables with no pivot in their columns are called **free variables** (x_3)

variables with a pivot in their columns are called **leading variables** (x_1, x_2, x_4)

Procedure: $Ax = b \rightsquigarrow (A \mid b) \overset{\text{Gaussian elimination}}{\underset{\text{row operations}}{\rightsquigarrow}} (A' \mid b')$ row echelon form

$\rightsquigarrow$ put free variable to the right-hand side $\rightsquigarrow$ backwards substitution $\rightsquigarrow$ solutions $\mathcal{S}$

Example:

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

free variables: x_2, x_5

$$\rightsquigarrow \left(\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \boxed{1} & 0 & 1 & 3-2x_2 \\ 0 & \boxed{2} & -1 & 2-4x_5 \\ 0 & 0 & \boxed{4} & 8-8x_5 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} \\ \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \\ \end{array}$$

III $4x_4 = 8-8x_5 \Rightarrow \boxed{x_4 = 2-2x_5}$ $\quad x_5 \in \mathbb{R}$

II $2x_3 - x_4 = 2-4x_5$

$\Rightarrow 2x_3 - 2 + 2x_5 = 2 - 4x_5 \Rightarrow 2x_3 = 4 - 6x_5 \Rightarrow \boxed{x_3 = 2-3x_5}$

I $x_1 + x_4 = 3 - 2x_2 \Rightarrow x_1 + 2 - 2x_5 = 3 - 2x_2 \Rightarrow \boxed{x_1 = 1 - 2x_2 + 2x_5}$

set of solutions:
$$S = \left\{ \begin{pmatrix} 1-2x_2+2x_5 \\ x_2 \\ 2-3x_5 \\ 2-2x_5 \\ x_5 \end{pmatrix} \,\middle|\, x_2, x_5 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \,\middle|\, x_2, x_5 \in \mathbb{R} \right\}$$