Linear Algebra - Part 40

Row echelon form

$$A = \begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & \boxed{4} & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: A matrix $A \in \mathbb{R}^{m \times n}$ is in row echelon form if:

(1) All zero rows (if there are any) are at the bottom.

(2) For each row: the first non-zero entry is strictly to the right of the first non-zero entry of the row above.

pivots

$$A = \begin{pmatrix} \boxed{1} & 3 & 5 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition:

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 5 & 0 & 1 \\ 0 & \boxed{2} & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

variables with no pivot in their columns are called free variables (x_3)

variables with a pivot in their columns are called leading variables (x_1, x_2, x_4)

Procedure: $Ax = b \rightsquigarrow (A \mid b) \xrightarrow[\text{row operations}]{\text{Gaussian elimination}} (A' \mid b')$ row echelon form

$\Downarrow$

solutions $\mathcal{S}$ $\leftarrow$ backwards substitution $\leftarrow$ put free variable to the right-hand side

Example:

free variables

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 - 2x_2 \\ 0 & \boxed{2} & -1 & 2 - 4x_5 \\ 0 & 0 & \boxed{4} & 8 - 8x_5 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \\ \end{array}$$

III $\quad 4x_4 = 8 - 8x_5 \quad \Rightarrow \quad \boxed{x_4 = 2 - 2x_5} \qquad x_5 \in \mathbb{R}$

II $\quad 2x_3 - x_4 = 2 - 4x_5$

$$\Rightarrow 2x_3 - 2 + 2x_5 = 2 - 4x_5 \Rightarrow 2x_3 = 4 - 6x_5 \Rightarrow \boxed{x_3 = 2 - 3x_5}$$

I $\quad x_1 + x_4 = 3 - 2x_2 \Rightarrow x_1 + 2 - 2x_5 = 3 - 2x_2 \Rightarrow \boxed{x_1 = 1 - 2x_2 + 2x_5}$

set of solutions:

$$S = \left\{ \begin{pmatrix} 1 - 2x_2 + 2x_5 \\ x_2 \\ 2 - 3x_5 \\ 2 - 2x_5 \\ x_5 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$