Linear Algebra - Part 39

Goal: Gaussian elimination (named after Carl Friedrich Gauß)

solve $Ax = b$

↳ use row operations to bring $(A|b)$ into upper triangular form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

backwards substitution:

third row: $3x_3 = 1 \Rightarrow \boxed{x_3 = \frac{1}{3}}$

second row: $2x_2 + x_3 = 1 \Rightarrow \boxed{x_2 = \frac{1}{3}}$

first row: $1x_1 + 2x_2 + 3x_3 = 1 \Rightarrow x_1 = -\frac{2}{3}$

↳ or use row operations to bring $(A|b)$ into row echelon form

↳ construct solution set

Example: system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 - 1x_3 &= 4 \\ 2x_1 - 1x_2 + 7x_3 &= 0 \\ 6x_1 + 13x_2 - 4x_3 &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 2 & -1 & 7 & 0 \\ 6 & 13 & -4 & 9 \end{array}\right) \begin{array}{l} \\ -1\cdot\mathrm{I} \\ -3\cdot\mathrm{I} \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 4 & -1 & -3 \end{array}\right) \begin{array}{l} \\ \\ +1\cdot\mathrm{II} \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 7 & -7 \end{array}\right) \xrightarrow[\text{substitution}]{\text{backwards}} \begin{array}{l} x_3 = -1 \\ x_2 = -1 \\ x_1 = 3 \end{array}$$

set of solutions: $S = \left\{ \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}$

Gaussian elimination:

$$\left(\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \cdots & a_{1n} & b_1 \\ \boxed{a_{21}} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \boxed{a_{m1}} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} - & \alpha_1^T & - \\ - & \alpha_2^T & - \\ & \vdots & \\ - & \alpha_m^T & - \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \alpha_1^T \\ \alpha_2^T - \frac{a_{21}}{a_{11}} \alpha_1^T \\ \vdots \\ \alpha_m^T - \frac{a_{m1}}{a_{11}} \alpha_1^T \end{pmatrix}$$ continue iteratively $\rightsquigarrow$ ... row echelon form