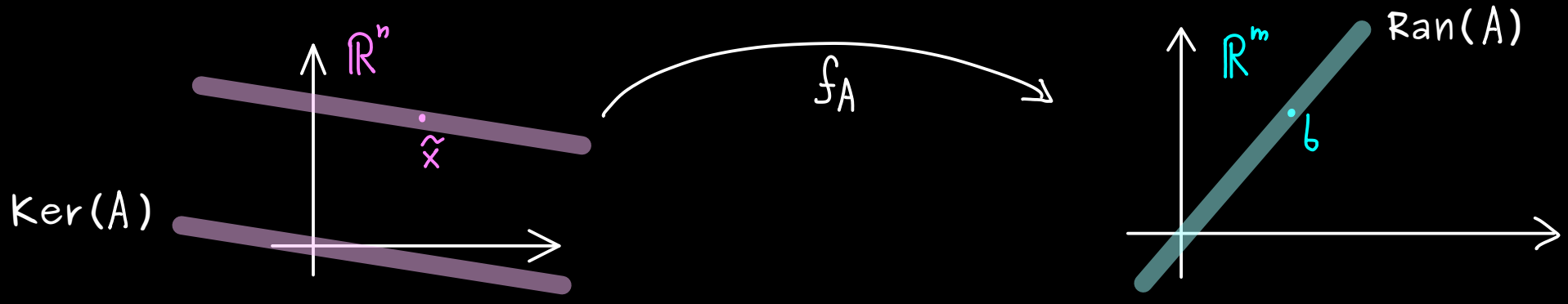
Linear Algebra – Part 38

Set of solutions: $Ax = b \qquad (A \in \mathbb{R}^{m \times n})$

solution: $\tilde{x}$ satisfies $A\tilde{x} = b$



uniqueness needs $\mathrm{Ker}(A) = \{0\}$

existence needs $b \in \mathrm{Ran}(A)$

Proposition: For a system $Ax = b \qquad (A \in \mathbb{R}^{m \times n})$

the set of solutions $S := \{ \tilde{x} \in \mathbb{R}^n \mid A\tilde{x} = b \}$

is an affine subspace (or empty).

More concretely: We have either $S = \emptyset$

or $S = V_0 + \mathrm{Ker}(A)$ for a vector $V_0 \in \mathbb{R}^n$

$= \{ V_0 + X_0 \mid X_0 \in \mathrm{Ker}(A) \}$

Proof: Assume $V_0 \in S. \Rightarrow AV_0 = b$

Set $\tilde{x} := V_0 + X_0$ for a vector $X_0 \in \mathbb{R}^n$.

Then: $\tilde{x} \in S \Leftrightarrow A\tilde{x} = b \Leftrightarrow AV_0 + AX_0 = b$

($\tilde{x} = V_0 + X_0$; $AV_0 = b$)

$\Leftrightarrow AX_0 = 0 \Leftrightarrow X_0 \in \mathrm{Ker}(A)$ □

Remember: Row operations don't change the set of solutions!

$S = V_0 + \mathrm{Ker}(A) = \mathrm{Ker}(MA)$

$AV_0 = b \Leftrightarrow MAV_0 = Mb$

⇝ Gaussian elimination
- decide $b \in \mathrm{Ran}(A)$
- gives us a particular solution V_0
- gives us $\mathrm{Ker}(A)$