Linear Algebra – Part 37

$$Ax = b \xrightarrow{\text{augmented matrix}} (A \mid b)$$

$$A \overset{\text{reversible manipulation}}{\longleftrightarrow} \tilde{A} : \qquad \underset{\text{invertible}}{M} A = \tilde{A} \longleftrightarrow A = M^{-1}\tilde{A}$$

For the system of linear equations: $\quad Ax = b \longleftrightarrow MAx = Mb \quad$ (new system)

Example: $\quad A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \rightsquigarrow MA = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{---} & \alpha_1^T & \text{---} \\ & \vdots & \\ \text{---} & \alpha_m^T & \text{---} \end{pmatrix}$$

$$c^T = (0,\dots,0,c_i,0,\dots,0,c_j,0,\dots,0) \Rightarrow c^T A = c_i\,\alpha_i^T + c_j\,\alpha_j^T$$

Example:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} \text{---} & \alpha_1^T & \text{---} \\ \text{---} & \alpha_2^T & \text{---} \\ \text{---} & \alpha_3^T & \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} & \alpha_1^T & \text{---} \\ \text{---} & \alpha_2^T & \text{---} \\ & \alpha_3^T + \lambda \cdot \alpha_1^T & \end{pmatrix}$$

invertible with inverse: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$

Definition: $\quad Z_{i+\lambda j} \in \mathbb{R}^{m\times m}, \quad i \neq j, \quad \lambda \in \mathbb{R},$

defined as the identity matrix with λ at the (i,j)th position.

Example: (exchanging rows)

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_{1\leftrightarrow 3}} \begin{pmatrix} — & \alpha_1^T & — \\ — & \alpha_2^T & — \\ — & \alpha_3^T & — \end{pmatrix} = \begin{pmatrix} — & \alpha_3^T & — \\ — & \alpha_2^T & — \\ — & \alpha_1^T & — \end{pmatrix}$$

Definition: $P_{i\leftrightarrow j} \in \mathbb{R}^{m\times m}$, $i \neq j$, defined as the identity matrix where the ith and the jth rows are exchanged.

Definition: (scaling rows)

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} — & \alpha_1^T & — \\ & \vdots & \\ — & \alpha_m^T & — \end{pmatrix} = \begin{pmatrix} — & d_1\,\alpha_1^T & — \\ & \vdots & \\ — & d_m\,\alpha_m^T & — \end{pmatrix}$$

with $d_k \neq 0$

Definition: row operations: finite combination of $Z_{i+\lambda j}$, $P_{i\leftrightarrow j}$, $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}$, ...

(for example: $M = Z_{3+7\,1}\, Z_{2+8\,1}\, P_{1\leftrightarrow 2}$)

Property: For $A \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{R}^{m\times m}$ (invertible), we have:

$$\mathrm{Ker}(MA) = \mathrm{Ker}(A) \quad , \quad \mathrm{Ran}(MA) = M\,\mathrm{Ran}(A)$$

$$\mathrel{:=} \{ My \mid y \in \mathrm{Ran}(A) \}$$