



## Linear Algebra - Part 37

$$Ax = b \xrightarrow{\text{augmented matrix}} (A|b)$$

$$A \xleftrightarrow{\text{reversible manipulation}} \tilde{A} : \begin{matrix} \text{invertible} \\ \uparrow \\ MA = \tilde{A} \end{matrix} \iff A = M^{-1}\tilde{A}$$

For the system of linear equations:

$$Ax = b \iff MAx = Mb \quad (\text{new system})$$

Example:  $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \rightsquigarrow MA = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{---} \alpha_1^T \text{---} \\ \vdots \\ \text{---} \alpha_m^T \text{---} \end{pmatrix}$$

$$c^T = (0, \dots, 0, c_i, 0, \dots, 0, c_j, 0, \dots, 0) \Rightarrow c^T A = c_i \alpha_i^T + c_j \alpha_j^T$$

Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} \begin{pmatrix} \text{---} \alpha_1^T \text{---} \\ \text{---} \alpha_2^T \text{---} \\ \text{---} \alpha_3^T \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} \alpha_1^T \text{---} \\ \text{---} \alpha_2^T \text{---} \\ \alpha_3^T + \lambda \cdot \alpha_1^T \end{pmatrix}$$

invertible with inverse:  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$

$Z_{3+\lambda 1}$

Definition:

$$Z_{i+\lambda j} \in \mathbb{R}^{m \times m}, \quad i \neq j, \quad \lambda \in \mathbb{R},$$

defined as the identity matrix with  $\lambda$  at the  $(i, j)$ th position.

Example: (exchanging rows)

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_{1 \leftrightarrow 3}} \begin{pmatrix} \text{---} \alpha_1^T \text{---} \\ \text{---} \alpha_2^T \text{---} \\ \text{---} \alpha_3^T \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} \alpha_3^T \text{---} \\ \text{---} \alpha_2^T \text{---} \\ \text{---} \alpha_1^T \text{---} \end{pmatrix}$$

Definition:  $P_{i \leftrightarrow j} \in \mathbb{R}^{m \times m}$ ,  $i \neq j$ , defined as the identity matrix where the  $i$ th and the  $j$ th rows are exchanged.

Definition: (scaling rows)

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} \text{---} \alpha_1^T \text{---} \\ \vdots \\ \text{---} \alpha_m^T \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} d_1 \alpha_1^T \text{---} \\ \vdots \\ \text{---} d_m \alpha_m^T \text{---} \end{pmatrix}$$

with  $d_k \neq 0$

Definition: row operations: finite combination of  $Z_{i+\lambda j}$ ,  $P_{i \leftrightarrow j}$ ,  $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}$ , ...  
 (for example:  $M = Z_{3+71} Z_{2+81} P_{1 \leftrightarrow 2}$ )

Property: For  $A \in \mathbb{R}^{m \times n}$  and  $M \in \mathbb{R}^{m \times m}$  (invertible), we have:

$$\text{Ker}(MA) = \text{Ker}(A), \quad \text{Ran}(MA) = M \text{Ran}(A)$$

$$\Leftrightarrow \{My \mid y \in \text{Ran}(A)\}$$