The Bright Side of Mathematics

Linear Algebra – Part 37

$$Ax = b \quad \overset{\text{augmented matrix}}{\rightsquigarrow} \quad (A \mid b)$$

$$A \overset{\text{reversible manipulation}}{\leftrightsquigarrow} \tilde{A} \; : \quad \underset{\text{invertible}}{M} A = \tilde{A} \;\leftrightsquigarrow\; A = M^{-1}\tilde{A}$$

For the system of linear equations: $\quad Ax = b \leftrightsquigarrow MAx = Mb \quad$ (new system)

Example: $\quad A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \rightsquigarrow MA = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{——} & \alpha_1^T & \text{——} \\ & \vdots & \\ \text{——} & \alpha_m^T & \text{——} \end{pmatrix}$$

$$c^T = (0, \dots, 0, c_i, 0, \dots, 0, c_j, 0, \dots, 0) \Rightarrow c^T A = c_i \alpha_i^T + c_j \alpha_j^T$$

Example:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} \text{——} & \alpha_1^T & \text{——} \\ \text{——} & \alpha_2^T & \text{——} \\ \text{——} & \alpha_3^T & \text{——} \end{pmatrix} = \begin{pmatrix} \text{——} & \alpha_1^T & \text{——} \\ \text{——} & \alpha_2^T & \text{——} \\ & \alpha_3^T + \lambda \cdot \alpha_1^T & \end{pmatrix}$$

invertible with inverse: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$

Definition: $\quad Z_{i+\lambda j} \in \mathbb{R}^{m \times m}, \; i \neq j, \; \lambda \in \mathbb{R},$

defined as the identity matrix with λ at the (i,j)th position.

Example: (exchanging rows)

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_{1 \leftrightarrow 3}} \begin{pmatrix} \text{——} & \alpha_1^T & \text{——} \\ \text{——} & \alpha_2^T & \text{——} \\ \text{——} & \alpha_3^T & \text{——} \end{pmatrix} = \begin{pmatrix} \text{——} & \alpha_3^T & \text{——} \\ \text{——} & \alpha_2^T & \text{——} \\ \text{——} & \alpha_1^T & \text{——} \end{pmatrix}$$

Definition: $\quad P_{i \leftrightarrow j} \in \mathbb{R}^{m \times m}, \; i \neq j,$ defined as the identity matrix where the ith and the jth rows are exchanged.

Definition: (scaling rows)

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} \text{——} & \alpha_1^T & \text{——} \\ & \vdots & \\ \text{——} & \alpha_m^T & \text{——} \end{pmatrix} = \begin{pmatrix} \text{——} & d_1 \alpha_1^T & \text{——} \\ & \vdots & \\ \text{——} & d_m \alpha_m^T & \text{——} \end{pmatrix}$$

with $d_k \neq 0$

Definition: row operations: finite combination of $Z_{i+\lambda j}, P_{i \leftrightarrow j}, \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}, \dots$

(for example: $M = Z_{3+71} \, Z_{2+81} \, P_{1 \leftrightarrow 2}$)

Property: For $A \in \mathbb{R}^{m \times n}$ and $M \in \mathbb{R}^{m \times m}$ (invertible), we have:

$$\mathrm{Ker}(MA) = \mathrm{Ker}(A) \quad , \quad \mathrm{Ran}(MA) = M\,\mathrm{Ran}(A) \overset{\text{``}}{=} \{ My \mid y \in \mathrm{Ran}(A) \}$$