Linear Algebra - Part 33

$A \in \mathbb{R}^{m \times n} \rightsquigarrow A^T \in \mathbb{R}^{n \times m}$

standard inner product in $\mathbb{R}^n \rightsquigarrow \langle u, v \rangle \in \mathbb{R}$
$= u^T v$

Proposition: For $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\langle y, Ax \rangle = \langle A^T y, x \rangle$$

inner product in $\mathbb{R}^m$ inner product in $\mathbb{R}^n$

Proof: $\langle \tilde{u}, \tilde{v} \rangle = \tilde{u}^T \tilde{v}$ for $\tilde{u}, \tilde{v} \in \mathbb{R}^m$

$(A^T y)^T = y^T \cdot (A^T)^T$

$$\langle y, \overbrace{Ax} \rangle = y^T (Ax) = (y^T A) x = (A^T y)^T x = \langle A^T y, x \rangle \qquad \square$$

Alternative definition: A^T is the only matrix $B \in \mathbb{R}^{n \times m}$ that satisfies:

$$\langle y, Ax \rangle = \langle By, x \rangle \quad \text{for all } x, y$$