Linear Algebra - Part 33

$$A \in \mathbb{R}^{m \times n} \rightsquigarrow A^T \in \mathbb{R}^{n \times m}$$

standard inner product in $\mathbb{R}^n \rightsquigarrow \langle u, v \rangle \in \mathbb{R}$

$$= u^T v$$

Proposition: For $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\underbrace{\langle y, Ax \rangle}_{\text{inner product in } \mathbb{R}^m} = \underbrace{\langle A^T y, x \rangle}_{\text{inner product in } \mathbb{R}^n}$$

Proof: $\langle \tilde{u}, \tilde{v} \rangle = \tilde{u}^T \tilde{v}$ for $\tilde{u}, \tilde{v} \in \mathbb{R}^m$

$$(A^T y)^T = y^T (A^T)^T$$

$$\langle y, \widetilde{Ax} \rangle = y^T (Ax) = (y^T A) x = (A^T y)^T x = \langle A^T y, x \rangle \qquad \square$$

Alternative definition: A^T is the only matrix $B \in \mathbb{R}^{n \times m}$ that satisfies:

$$\langle y, Ax \rangle = \langle By, x \rangle \quad \text{for all } x, y$$