Linear Algebra – Part 31

matrices $A, B \in \mathbb{R}^{n \times n}$

$$\begin{tikzcd} \bigcirc \arrow[r, bend left, "f_B"] \arrow[rr, bend right, "f_{AB} = f_A \circ f_B"'] & \bigcirc \arrow[l, bend left, "f_{B^{-1}}"] \arrow[r, bend left, "f_A"] & \bigcirc \arrow[l, bend left, "f_{A^{-1}}"] \end{tikzcd}$$

We have: $f_{B^{-1}} \circ f_{A^{-1}} = (f_{AB})^{-1} \Rightarrow \boxed{(AB)^{-1} = B^{-1}A^{-1}}$

Important fact: $f: \mathbb{R}^n \to \mathbb{R}^n$ linear and bijective

$\Rightarrow f^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is also linear

Proof: $f^{-1}(\lambda y) = f^{-1}(\lambda \cdot f(x)) \underbrace{=}_{f \text{ linear}} f^{-1}(f(\lambda x)) = \lambda \cdot x = \lambda f^{-1}(y)$ ✓

There is exactly one x with $f(x) = y$

$f^{-1}(y + \tilde{y}) = f^{-1}(f(x) + f(\tilde{x})) \underbrace{=}_{f \text{ linear}} f^{-1}(f(x + \tilde{x})) = x + \tilde{x}$

$= f^{-1}(y) + f^{-1}(\tilde{y})$ ✓