Linear Algebra – Part 30

injectivity, surjectivity, bijectivity for square matrices

system of linear equations: $Ax = b \overset{\text{if } A \text{ invertible}}{\Longrightarrow} A^{-1}Ax = A^{-1}b \Longrightarrow x = A^{-1}b$

<u>Theorem:</u> $A \in \mathbb{R}^{n \times n}$ square matrix. $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induced linear map.

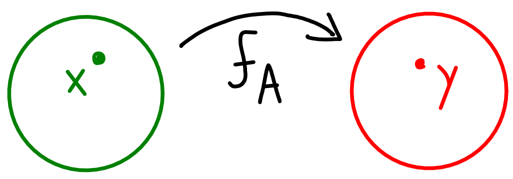
Then: f_A is injective $\Longleftrightarrow$ f_A is surjective

<u>Proof:</u> ($\Rightarrow$) f_A injective , standard basis of $\mathbb{R}^n$ $(e_1, \dots, e_n)$

$\Longrightarrow \underbrace{(f_A(e_1), \dots, f_A(e_n))}_{\text{basis of } \mathbb{R}^n}$ still linearly independent

$\Longrightarrow f_A$ is surjective

($\Leftarrow$) f_A surjective



For each $y \in \mathbb{R}^n$, you find $x \in \mathbb{R}^n$ with $f_A(x) = y$.

We know: $x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$

$y = f_A(x) = x_1 f_A(e_1) + x_2 f_A(e_2) + \dots + x_n f_A(e_n)$

$\Longrightarrow (f_A(e_1), \dots, f_A(e_n))$ spans $\mathbb{R}^n$

$\overset{n \text{ vectors}}{\Longrightarrow} (f_A(e_1), \dots, f_A(e_n))$ linearly independent

Assume $f_A(x) = f_A(\tilde{x}) \Longrightarrow f_A(\underbrace{x - \tilde{x}}_{v}) = 0$

$\Longrightarrow v_1 f_A(e_1) + v_2 f_A(e_2) + \dots + v_n f_A(e_n) = 0$

$\overset{\text{lin. independence}}{\Longrightarrow} v_1 = v_2 = \dots = v_n = 0$

$\Longrightarrow x = \tilde{x} \Longrightarrow f_A$ is injective $\square$