Linear Algebra - Part 30

injectivity, surjectivity, bijectivity for square matrices

system of linear equations: $Ax = b \overset{\text{if } A \text{ invertible}}{\Longrightarrow} A^{-1}Ax = A^{-1}b \Longrightarrow x = A^{-1}b$

Theorem: $A \in \mathbb{R}^{n \times n}$ square matrix. $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induced linear map.

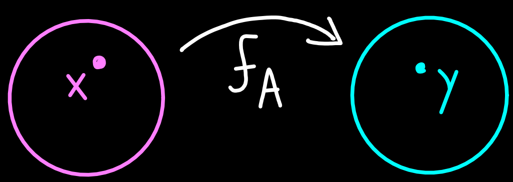
Then: f_A is injective $\Longleftrightarrow$ f_A is surjective

Proof: $(\Rightarrow)$ f_A injective, standard basis of $\mathbb{R}^n$ $(e_1, \dots, e_n)$

$$\Rightarrow \underbrace{\big(f_A(e_1), \dots, f_A(e_n)\big)}_{\text{basis of } \mathbb{R}^n} \text{ still linearly independent}$$

$$\Rightarrow f_A \text{ is surjective}$$

$(\Leftarrow)$ f_A surjective



For each $y \in \mathbb{R}^n$, you find $x \in \mathbb{R}^n$ with $f_A(x) = y$.

We know: $x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$

$$y = f_A(x) = x_1 f_A(e_1) + x_2 f_A(e_2) + \dots + x_n f_A(e_n)$$

$$\Rightarrow \big(f_A(e_1), \dots, f_A(e_n)\big) \text{ spans } \mathbb{R}^n$$

$$\overset{n \text{ vectors}}{\Longrightarrow} \big(f_A(e_1), \dots, f_A(e_n)\big) \text{ linearly independent}$$

Assume $f_A(x) = f_A(\tilde{x}) \Rightarrow f_A(\underbrace{x - \tilde{x}}_{v}) = 0$

$$\Rightarrow v_1 f_A(e_1) + v_2 f_A(e_2) + \dots + v_n f_A(e_n) = 0$$

$$\overset{\text{lin. independence}}{\Longrightarrow} v_1 = v_2 = \dots = v_n = 0$$

$$\Rightarrow x = \tilde{x} \Rightarrow f_A \text{ is injective} \qquad \square$$