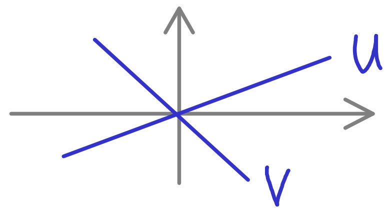
Linear Algebra - Part 28

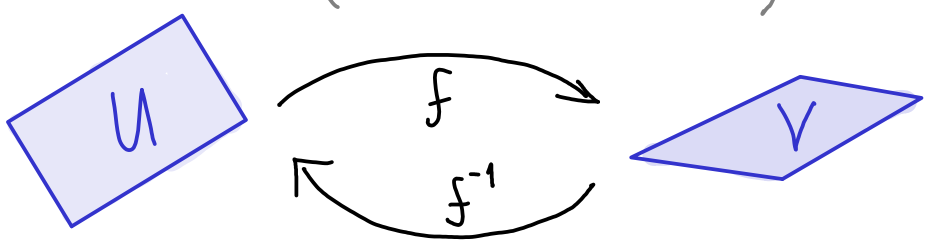
Dimension of U: number of elements in a basis of U $= \dim(U)$

Theorem: $U, V \subseteq \mathbb{R}^n$ linear subspaces



(a) $\dim(U) = \dim(V) \iff$ there is a bijective linear map $f: U \to V$

$\left(f^{-1}: V \to U \text{ linear} \right)$



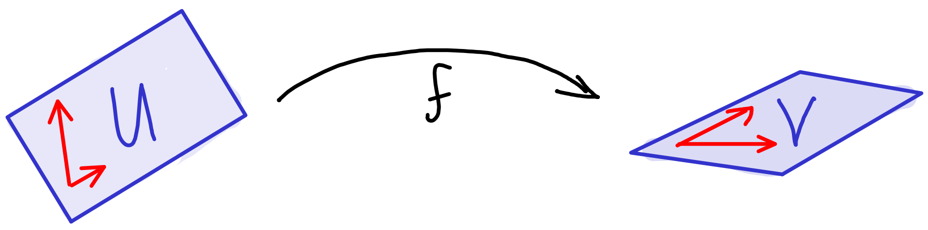
(b) $U \subseteq V$ and $\dim(U) = \dim(V) \implies U = V$

Proof: (a) $(\Rightarrow)$ We assume $\dim(U) = \dim(V)$.

Hence: $\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ basis of U

$\mathcal{C} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ basis of V

define: $f: U \to V$, $f(u^{(i)}) = v^{(i)}$



For $x \in U$: $f(x) = f\left(\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)}\right)$ uniquely determined $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

$$= \lambda_1 \cdot f(u^{(1)}) + \lambda_2 \cdot f(u^{(2)}) + \dots + \lambda_k \cdot f(u^{(k)})$$

$$= \lambda_1 \cdot v^{(1)} + \dots + \lambda_k \cdot v^{(k)} \quad =: f(x)$$

Now define: $f^{-1}: V \to U$, $f^{-1}(v^{(i)}) = u^{(i)}$

Then: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y \implies f$ is bijective+linear

$(\Leftarrow)$ We assume that there is $\underbrace{\text{bijective}}_{\text{injective+surjective}}$ linear map $f: U \to V$.

Let $\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ be a basis of U

$$\Longrightarrow \quad \left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)})\right) \quad \text{basis in } V \text{ ?}$$

f injective $\qquad$ f surjective

linearly independent $\qquad$ $\mathrm{Span}\left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)})\right) = V$

$$\Longrightarrow \quad \dim(U) = \dim(V)$$

(b) We show: $\boxed{U \subseteq V \text{ and } \dim(U) = \dim(V) \quad \Longrightarrow \quad U = V}$

$$(u^{(1)}, u^{(2)}, \dots, u^{(k)}) \text{ basis of } U \quad \Longrightarrow \quad (u^{(1)}, u^{(2)}, \dots, u^{(k)}) \text{ basis of } V$$

$$v = \lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)} \in U$$

$$\Longrightarrow \quad U = V$$

$\square$