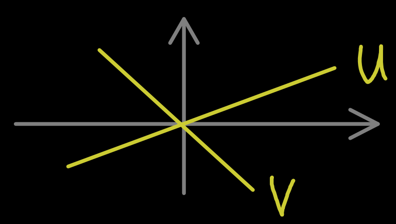
Linear Algebra – Part 28

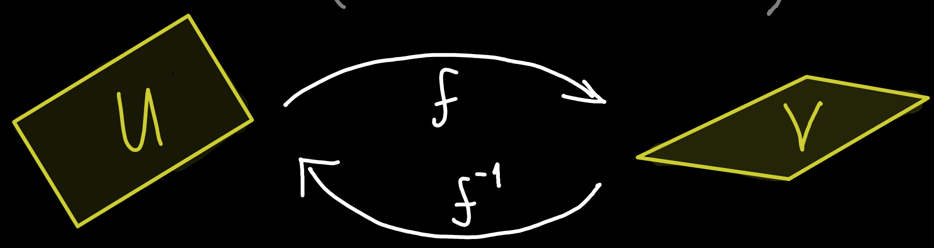
Dimension of U: number of elements in a basis of U $= \dim(U)$

Theorem: $U, V \subseteq \mathbb{R}^n$ linear subspaces



(a) $\dim(U) = \dim(V) \iff$ there is a bijective linear map $f: U \to V$

$\left(f^{-1}: V \to U \text{ linear} \right)$



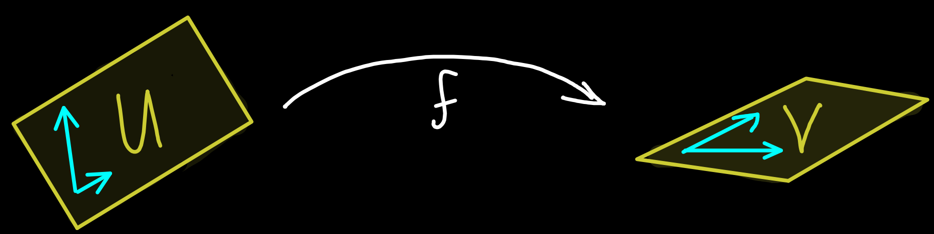
(b) $U \subseteq V$ and $\dim(U) = \dim(V) \implies U = V$

Proof: (a) $(\Rightarrow)$ We assume $\dim(U) = \dim(V)$.

Hence:

$$\begin{array}{ll} \mathcal{B} = \left(u^{(1)}, u^{(2)}, \dots, u^{(k)} \right) & \text{basis of } U \\ \quad\ \ \downarrow \qquad \downarrow \qquad \dots \quad \downarrow & \\ \mathcal{C} = \left(v^{(1)}, v^{(2)}, \dots, v^{(k)} \right) & \text{basis of } V \end{array}$$

define: $f: U \to V$, $f(u^{(i)}) = v^{(i)}$



For $x \in U$: $f(x) = f\left(\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)} \right)$ uniquely determined $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

$$= \lambda_1 \cdot f\left(u^{(1)}\right) + \lambda_2 \cdot f\left(u^{(2)}\right) + \dots + \lambda_k \cdot f\left(u^{(k)}\right)$$

$$= \lambda_1 \cdot v^{(1)} + \dots + \lambda_k \cdot v^{(k)} =: f(x)$$

Now define: $f^{-1}: V \to U$, $f^{-1}(v^{(i)}) = u^{(i)}$

Then: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y \implies f$ is bijective+linear

($\Leftarrow$) We assume that there is $\underbrace{\text{bijective}}_{\text{injective+surjective}}$ linear map $f: U \to V$.

Let $\mathcal{B} = \left(u^{(1)}, u^{(2)}, \dots, u^{(k)}\right)$ be a basis of U

$\Longrightarrow \left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)}) \right)$ basis in V ?

f injective: linearly independent

f surjective: $\operatorname{Span}\left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)})\right) = V$

$$\Longrightarrow \dim(U) = \dim(V)$$

(b) We show: $\boxed{U \subseteq V \text{ and } \dim(U) = \dim(V) \Longrightarrow U = V}$

$\left(u^{(1)}, u^{(2)}, \dots, u^{(k)}\right)$ basis of U $\Longrightarrow$ $\left(u^{(1)}, u^{(2)}, \dots, u^{(k)}\right)$ basis of V

$$v = \lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)} \in U$$

$$\Longrightarrow U = V$$

$\square$