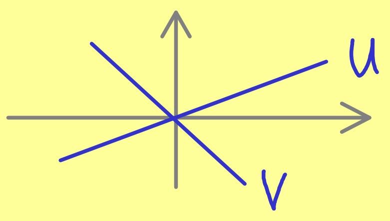
The Bright Side of Mathematics

Linear Algebra – Part 28

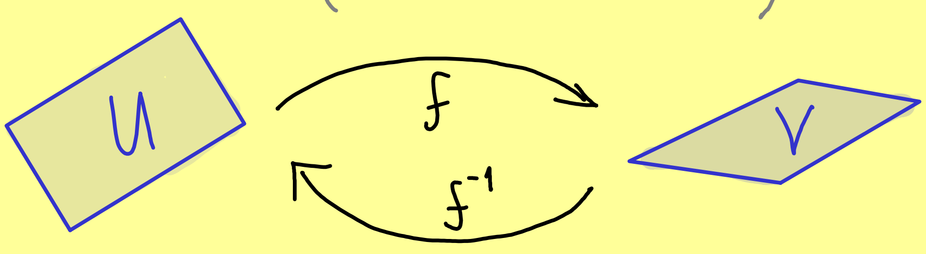
Dimension of $\mathcal{U}$: number of elements in a basis of $\mathcal{U}$ $= \dim(\mathcal{U})$

Theorem: $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^n$ linear subspaces



(a) $\dim(\mathcal{U}) = \dim(\mathcal{V}) \iff$ there is a bijective linear map $f: \mathcal{U} \to \mathcal{V}$

$\big(f^{-1}: \mathcal{V} \to \mathcal{U}$ linear $\big)$



(b) $\mathcal{U} \subseteq \mathcal{V}$ and $\dim(\mathcal{U}) = \dim(\mathcal{V}) \implies \mathcal{U} = \mathcal{V}$

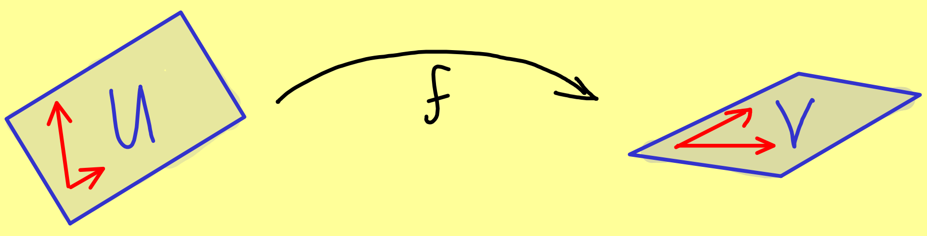
Proof: (a) $(\Rightarrow)$ We assume $\dim(\mathcal{U}) = \dim(\mathcal{V})$.

Hence:

$$\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)}) \quad \text{basis of } \mathcal{U}$$

$$\mathcal{C} = (v^{(1)}, v^{(2)}, \dots, v^{(k)}) \quad \text{basis of } \mathcal{V}$$

define: $f: \mathcal{U} \to \mathcal{V}$, $f(u^{(i)}) = v^{(i)}$



For $x \in \mathcal{U}$: $f(x) = f(\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)})$ uniquely determined $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

$$= \lambda_1 \cdot f(u^{(1)}) + \lambda_2 \cdot f(u^{(2)}) + \dots + \lambda_k \cdot f(u^{(k)})$$

$$= \lambda_1 \cdot v^{(1)} + \dots + \lambda_k \cdot v^{(k)} \quad =: f(x)$$

Now define: $f^{-1}: \mathcal{V} \to \mathcal{U}$, $f^{-1}(v^{(i)}) = u^{(i)}$

Then: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y \implies$ f is bijective+linear

$(\Leftarrow)$ We assume that there is $\underbrace{\text{bijective}}_{\text{injective+surjective}}$ linear map $f: \mathcal{U} \to \mathcal{V}$.

Let $\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ be a basis of $\mathcal{U}$

$\implies \big(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)}) \big)$ basis in $\mathcal{V}$?

f injective f surjective

linearly independent $\quad \operatorname{Span}\big(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)})\big) = \mathcal{V}$

$$\implies \dim(\mathcal{U}) = \dim(\mathcal{V})$$

(b) We show: $\boxed{\mathcal{U} \subseteq \mathcal{V} \text{ and } \dim(\mathcal{U}) = \dim(\mathcal{V}) \implies \mathcal{U} = \mathcal{V}}$

$(u^{(1)}, u^{(2)}, \dots, u^{(k)})$ basis of $\mathcal{U} \implies (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ basis of $\mathcal{V}$

$$v = \lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)} \in \mathcal{U}$$

$$\implies \mathcal{U} = \mathcal{V} \qquad \square$$