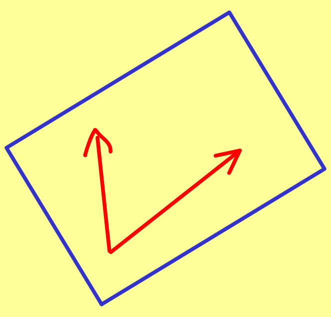
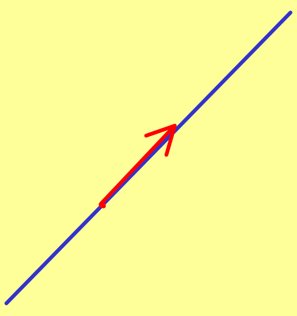
The Bright Side of Mathematics

Linear Algebra – Part 26



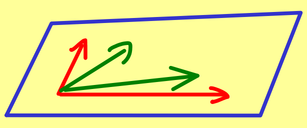
dimension = 2



dimension = 1

Steinitz Exchange Lemma

Let $U \subseteq \mathbb{R}^n$ be a subspace and



$\mathcal{B} = \left(v^{(1)}, v^{(2)}, \dots, v^{(k)}\right)$ be a basis of U.

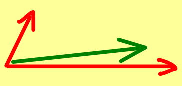
$\mathcal{A} = \left(a^{(1)}, a^{(2)}, \dots, a^{(\ell)}\right)$ linearly independent vectors in U.

Then: One can add $k-\ell$ vectors from $\mathcal{B}$ to the family $\mathcal{A}$ such that we get a new basis of U.

Proof: $\ell = 1$: $\mathcal{B} \cup \mathcal{A} = \left(v^{(1)}, v^{(2)}, \dots, v^{(k)}, a^{(1)}\right)$ is linearly dependent

because $\mathcal{B}$ is a basis: there are uniquely given $\lambda_1, \dots, \lambda_k \in \mathbb{R}$:

$$(*) \qquad a^{(1)} = \lambda_1 v^{(1)} + \cdots + \lambda_k v^{(k)}$$



Choose $\lambda_j \neq 0$:

$$v^{(j)} = \frac{1}{-\lambda_j}\left(\lambda_1 v^{(1)} + \cdots + \lambda_{j-1} v^{(j-1)} + \lambda_{j+1} v^{(j+1)} + \cdots + \lambda_k v^{(k)} - a^{(1)}\right)$$

Remove $v^{(j)}$ from $\mathcal{B} \cup \mathcal{A}$ and call it $\mathcal{C}$.

$\mathcal{C}$ is linearly independent:

$$\tilde{\lambda}_1 v^{(1)} + \cdots + \tilde{\lambda}_{j-1} v^{(j-1)} + \tilde{\lambda}_j a^{(1)} + \tilde{\lambda}_{j+1} v^{(j+1)} + \cdots + \tilde{\lambda}_k v^{(k)} = 0$$

Assume $\tilde{\lambda}_j \neq 0$: $a^{(1)} =$ linear combination with $v^{(1)}, \dots, v^{(j-1)}, v^{(j+1)}, \dots, v^{(k)}$ $\lightning$ $(*)$

Hence: $\tilde{\lambda}_j = 0 \implies$

$$\tilde{\lambda}_1 v^{(1)} + \cdots + \tilde{\lambda}_{j-1} v^{(j-1)} + \tilde{\lambda}_{j+1} v^{(j+1)} + \cdots + \tilde{\lambda}_k v^{(k)} = 0$$

$$\overset{\text{lin. independence}}{\Longrightarrow} \quad \tilde{\lambda}_i = 0 \quad \text{for } i \in \{1, \dots, k\}$$

$\mathcal{C}$ spans U : $u \in U \overset{\mathcal{B}\text{ basis}}{\Longrightarrow}$ there are coefficients

$$\boxed{v^{(j)} = \frac{1}{-\lambda_j}\left(\lambda_1 v^{(1)} + \cdots + \lambda_{j-1} v^{(j-1)} + \lambda_{j+1} v^{(j+1)} + \cdots + \lambda_k v^{(k)} - a^{(1)}\right)}$$

$$u = \mu_1 v^{(1)} + \cdots + \mu_{j-1} v^{(j-1)} + \mu_j v^{(j)} + \mu_{j+1} v^{(j+1)} + \cdots + \mu_k v^{(k)}$$

$$= \tilde{\mu}_1 v^{(1)} + \cdots + \tilde{\mu}_{j-1} v^{(j-1)} + \tilde{\mu}_j a^{(1)} + \tilde{\mu}_{j+1} v^{(j+1)} + \cdots + \tilde{\mu}_k v^{(k)}$$