Linear Algebra – Part 20

Linear map: $f: \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto f(x)$

n components

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

canonical unit vectors

$$\left.\begin{aligned} f(x) &= f(x_1 e_1 + x_2 e_2 + \dots + x_n e_n) \\ &\overset{\text{linearity}}{=} x_1 f(e_1) + x_2 f(e_2) + \dots + x_n f(e_n) \end{aligned}\right\} \Longrightarrow$$

to know $f(x)$, it's sufficient to know $f(e_1), \dots, f(e_n)$

<u>Proposition:</u> $f: \mathbb{R}^n \to \mathbb{R}^m$ linear.

Then there is exactly one matrix $A \in \mathbb{R}^{m \times n}$ with $f = f_A$ $\quad (f(x) = Ax)$

and

$$A = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}.$$

<u>Proof:</u>

$$f_A(x) = f_A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ f(e_1) \\ | \end{pmatrix} + \dots + x_n \begin{pmatrix} | \\ f(e_n) \\ | \end{pmatrix}$$

$$= f(x)$$

Uniqueness: Assume there are $A, B \in \mathbb{R}^{m \times n}$ with $f = f_A$ and $f = f_B$

$\Longrightarrow \quad Ax = Bx \quad$ for all $x \in \mathbb{R}^n$

$\Longrightarrow \quad (A - B)x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad$ for all $x \in \mathbb{R}^n$

Use e_i

$\Longrightarrow \quad A - B = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \Longrightarrow \quad A = B \qquad \square$