Linear Algebra – Part 19

$$A \in \mathbb{R}^{m \times n} \rightsquigarrow f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longmapsto Ax$$

Proposition: f_A is a linear map:

(1) $f_A(x+y) = f_A(x) + f_A(y)$, $A(x+y) = Ax + Ay$ (distributive)

(2) $f_A(\lambda \cdot x) = \lambda \cdot f_A(x)$, $A(\lambda \cdot x) = \lambda \cdot (Ax)$ (compatible)

Example:

$$\begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

$$= \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} (x_1 + y_1) + \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} (x_2 + y_2)$$

$$= \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} x_1 + \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} x_2 + \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} y_1 + \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} y_2$$

$$= \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

matrix A (table of numbers) $\leftrightsquigarrow$ f_A abstract linear map

Now: two matrices A, B

$$\left. \begin{array}{l} A \in \mathbb{R}^{m \times k} \\ B \in \mathbb{R}^{k \times n} \end{array} \right\} AB \in \mathbb{R}^{m \times n} \quad \leftrightsquigarrow$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "f_B"] \arrow[rr, bend right, "f_A \circ f_B"'] & \mathbb{R}^k \arrow[r, bend left, "f_A"] & \mathbb{R}^m \end{tikzcd}$$

$$\underbrace{(f_A \circ f_B)}_{f_{AB}}(x) = f_A(f_B(x)) = f_A(Bx) = A(Bx) = (AB)x$$