Linear Algebra - Part 17

matrix product: $\mathbb{R}^{m\times n} \times \mathbb{R}^{n\times k} \longrightarrow \mathbb{R}^{m\times k}$

$$(A,B) \longmapsto AB$$

defined by: $$(AB)_{ij} = \sum_{\ell=1}^{n} a_{i\ell}\, b_{\ell j}$$

Properties: (a) $$(A+B)C = AC + BC$$
$$D(A+B) = DA + DB$$ (distributive laws)

(b) $$\lambda\cdot(AB) = (\lambda\cdot A)B = A(\lambda\cdot B)$$

(c) $$(AB)C = A(BC)$$ (associative law)

Proof: (c) $$\big((AB)C\big)_{ij} = \sum_{\ell=1}^{n} (AB)_{i\ell}\, c_{\ell j}$$

$$= \sum_{\ell}\Big(\sum_{z} a_{iz} b_{z\ell}\Big) c_{\ell j}$$

$$= \sum_{z} a_{iz} \sum_{\ell} b_{z\ell}\, c_{\ell j} = \sum_{z} a_{iz} (BC)_{zj}$$

$$= \big(A(BC)\big)_{ij}$$

Important: no commutative law (in general)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\neq$$

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