Linear Algebra – Part 10

Cross product / vector product

↳ only $\mathbb{R}^3$

map $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

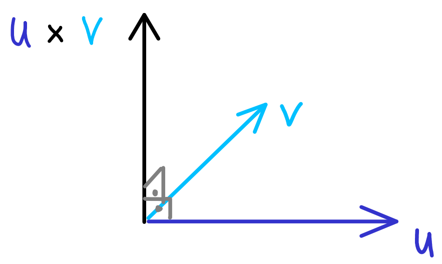
Definition: For $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$, we define the cross product:

$$u \times v = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

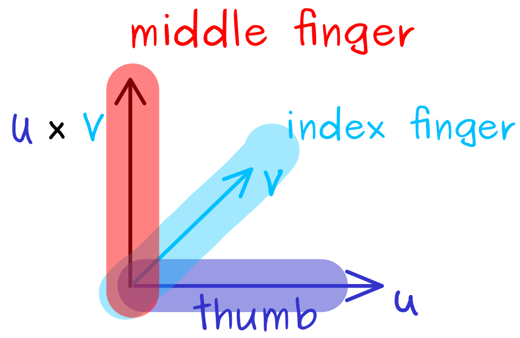
With Levi-Civita symbol: $u \times v = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, u_i\, v_j\, \underbrace{e_k}_{\text{canonical unit vector}}$

Properties:

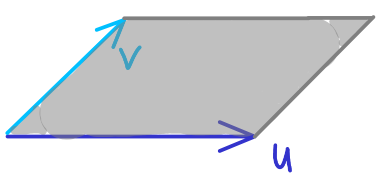
(1) orthogonality: $u \times v$ orthogonal to u
$u \times v$ orthogonal to v (with respect to the standard inner product)



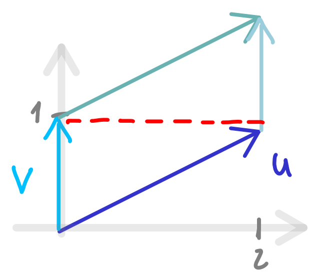
(2) orientation: right-hand rule



(3) length: $\|u \times v\|$ = area of the parallelogram



Example: $u = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$



$$u \times v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 0 - 0\cdot 1 \\ 0\cdot 0 - 2\cdot 0 \\ 2\cdot 1 - 1\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

(1) orthogonality ✓
(2) right-hand rule ✓
(3) length ✓