Linear Algebra – Part 7

Examples for subspaces: (1) $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = x_2 \text{ and } x_3 = -2x_2 \right\}$

Is this a subspace?

Checking: (a) Is the zero vector in U?

$$x = 0 \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{matrix} x_1 = 0 = x_2 \\ x_3 = 0 = -2x_2 \end{matrix}$$

$$\implies 0 \in U \checkmark$$

(b) Is U closed under scalar multiplication?

Assume: $u \in U$, $\lambda \in \mathbb{R}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

Then: $u_1 = u_2$
$u_3 = -2u_2$

What about? $x := \lambda \cdot u$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix}$

Do we have? $\begin{matrix} x_1 = x_2 \\ x_3 = -2x_2 \end{matrix}$ which is equivalent to $\begin{matrix} \lambda u_1 = \lambda u_2 \\ \lambda u_3 = -2 \cdot (\lambda u_2) \end{matrix}$

Proof: $\begin{matrix} u_1 = u_2 \\ u_3 = -2u_2 \end{matrix} \overset{\lambda \cdot}{\implies} \begin{matrix} \lambda u_1 = \lambda u_2 \\ \lambda u_3 = -2(\lambda u_2) \end{matrix} \implies x := \lambda \cdot u \in U \checkmark$

(c) Is U closed under vector addition?

Assume: $u, v \in U$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Then: $\begin{matrix} u_1 = u_2 \\ u_3 = -2u_2 \end{matrix}$ and $\begin{matrix} v_1 = v_2 \\ v_3 = -2v_2 \end{matrix}$

What about? $x := u + v$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$

Do we have? $\begin{matrix} x_1 = x_2 \\ x_3 = -2x_2 \end{matrix}$ which is equivalent to $\begin{matrix} u_1 + v_1 = u_2 + v_2 \\ u_3 + v_3 = -2(u_2 + v_2) \end{matrix}$

Proof: $\begin{matrix} u_1 = u_2 \\ u_3 = -2u_2 \end{matrix}$ and $\begin{matrix} v_1 = v_2 \\ v_3 = -2v_2 \end{matrix}$

$\Rightarrow \begin{matrix} u_1 + v_1 = u_2 + v_2 \\ u_3 + v_3 = -2u_2 + (-2v_2) \end{matrix} \Rightarrow \begin{matrix} u_1 + v_1 = u_2 + v_2 \\ u_3 + v_3 = -2(u_2 + v_2) \end{matrix}$

$\Rightarrow x := u + v \in U$ ✓

$\Rightarrow U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = x_2 \text{ and } x_3 = -2x_2 \right\}$ subspace!

(2) $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1^2 = x_2 \right\}$

Show that (b) does not hold: $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in U$, $\lambda = 2$

What about? $x := \lambda \cdot u = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin U$

$4 = 2^2 = x_1^2 \neq x_2 = 2$ $\Rightarrow$ not a subspace!