The Bright Side of Mathematics

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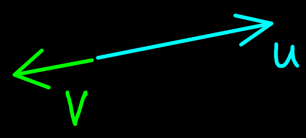
Have fun learning mathematics!

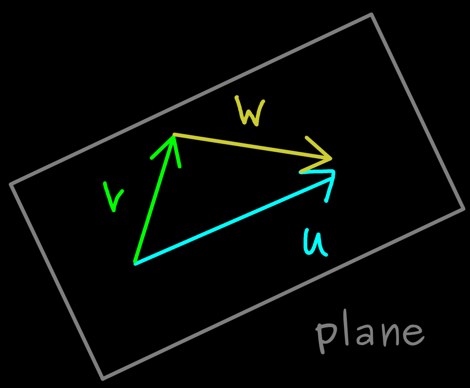
Linear Algebra - Part 22

$\mathbb{R}^2$:



colinear: $u = \lambda v$

$\mathbb{R}^3$:



coplanar: $u = \lambda v + \mu w$

$\iff 0 = (-1) u + \lambda v + \mu w$

<u>Definition:</u> Let $v^{(1)}, v^{(2)}, \ldots, v^{(k)} \in \mathbb{R}^n$. The family $\left(v^{(1)}, v^{(2)}, \ldots, v^{(k)}\right)$ (or $\{v^{(1)}, v^{(2)}, \ldots, v^{(k)}\}$) is called <u>linearly dependent</u> if there are $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ that are not all equal to zero such that:

$$\sum_{j=1}^{k} \lambda_j v^{(j)} = 0$$

zero vector in $\mathbb{R}^n$

We call the family <u>linearly independent</u> if

$$\sum_{j=1}^{k} \lambda_j v^{(j)} = 0 \implies \lambda_1 = \lambda_2 = \lambda_3 = \cdots = 0$$

Linear Algebra – Part 23

$(v^{(1)}, v^{(2)}, \dots, v^{(k)})$ linearly independent if

$$\sum_{j=1}^{k} \lambda_j v^{(j)} = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \cdots = 0$$

Examples: (a) $(v^{(1)})$ linearly independent if $v^{(1)} \neq 0$

(b) $(0, v^{(2)}, \dots, v^{(k)})$ linearly dependent

$$(\lambda_1 = 1 \quad , \lambda_2 = \lambda_3 = \cdots = 0)$$

(c) $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ linearly dependent

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

(d) $(e_1, e_2, \dots, e_n)$, $e_i \in \mathbb{R}^n$ canonical unit vectors

linearly independent

$$\sum_{j=1}^{n} \lambda_j e_j = 0 \iff \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \iff \lambda_1 = \lambda_2 = \lambda_3 = \cdots = 0$$

(e) $(e_1, e_2, \dots, e_n, v)$, $e_i, v \in \mathbb{R}^n$

linearly dependent

Fact: $(v^{(1)}, v^{(2)}, \dots, v^{(k)})$ family of vectors $v^{(j)} \in \mathbb{R}^n$

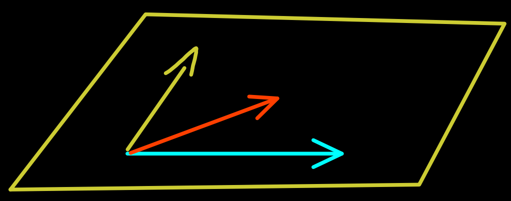
linearly dependent

$\iff$ There is ℓ with

$$\operatorname{Span}(v^{(1)}, v^{(2)}, \dots, v^{(k)}) = \operatorname{Span}(v^{(1)}, \dots, v^{(\ell-1)}, v^{(\ell+1)}, \dots, v^{(k)})$$

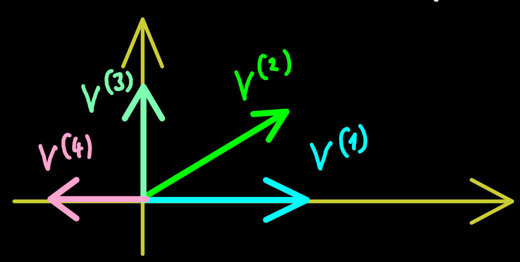
Linear Algebra - Part 24

subspace:



$U \subseteq \mathbb{R}^n$ with
(a) $0 \in U$
(b) $u \in U, \lambda \in \mathbb{R} \Rightarrow \lambda \cdot u \in U$
(c) $u, v \in U \Rightarrow u + v \in U$

plane: $\mathbb{R}^2$



$$\mathrm{Span}(v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}) = \mathbb{R}^2$$

$$\mathrm{Span}(v^{(1)}, v^{(3)}) = \mathbb{R}^2$$

$$\mathrm{Span}(v^{(1)}, v^{(4)}) = \mathbb{R} \times \{0\} \neq \mathbb{R}^2$$

Definition: $U \subseteq \mathbb{R}^n$ subspace, $\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$, $v^{(j)} \in \mathbb{R}^n$.

$\mathcal{B}$ is called a basis of U if:

(a) $U = \mathrm{Span}(\mathcal{B})$

(b) $\mathcal{B}$ is linearly independent

Example:

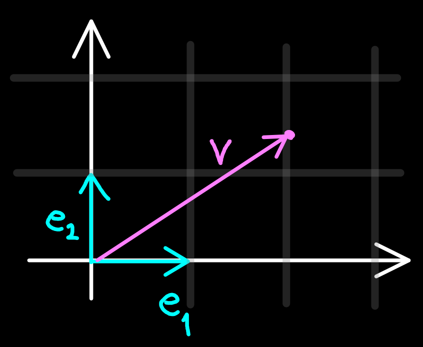
$$\mathbb{R}^n = \mathrm{Span}(\underbrace{e_1, \dots, e_n}_{\text{standard basis of } \mathbb{R}^n})$$

$$\mathbb{R}^3 = \mathrm{Span}\Bigg(\underbrace{\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\text{basis of } \mathbb{R}^3}\Bigg)$$

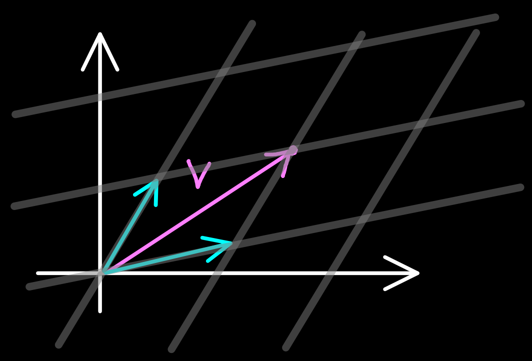
Linear Algebra - Part 25

basis of a subspace: spans the subspace + linearly independent

$\mathbb{R}^2$



$v = \begin{pmatrix} 2 \\ \frac{4}{3} \end{pmatrix}$



coordinates of v: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

coordinates: $U \subseteq \mathbb{R}^n$ subspace, $\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ basis of U

$\Longrightarrow$ Each vector $u \in U$ can be written as a linear combination:

$$u = \lambda_1 v^{(1)} + \lambda_2 v^{(2)} + \dots + \lambda_k v^{(k)} \quad , \quad \lambda_j \in \mathbb{R}$$

(uniquely determined)

coordinates of u with respect to $\mathcal{B}$

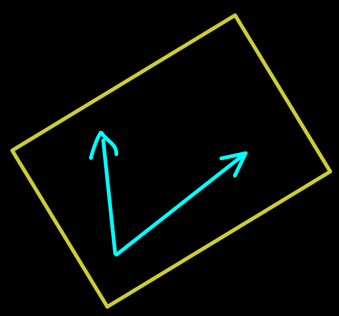
$$u = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}_{\mathcal{B}}$$

Example: $\mathbb{R}^3 = \operatorname{Span}\Bigg(\underbrace{\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\text{basis of } \mathbb{R}^3}\Bigg)$

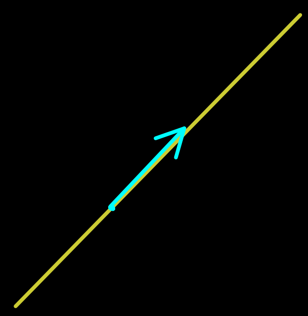
$$u = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$\tilde{u} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = -1 \cdot \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

Linear Algebra - Part 26



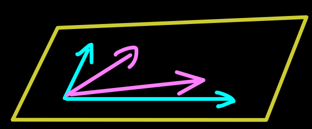
dimension = 2



dimension = 1

Steinitz Exchange Lemma

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a subspace and



$\mathcal{B} = \left(v^{(1)}, v^{(2)}, \dots, v^{(k)} \right)$ be a basis of $\mathcal{U}$.

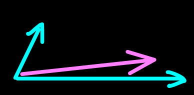
$\mathcal{A} = \left(a^{(1)}, a^{(2)}, \dots, a^{(\ell)} \right)$ linearly independent vectors in $\mathcal{U}$.

Then: One can add $k-\ell$ vectors from $\mathcal{B}$ to the family $\mathcal{A}$ such that we get a new basis of $\mathcal{U}$.

Proof: $\ell = 1$: $\mathcal{B} \cup \mathcal{A} = \left(v^{(1)}, v^{(2)}, \dots, v^{(k)}, a^{(1)} \right)$ is linearly dependent

because $\mathcal{B}$ is a basis: there are uniquely given $\lambda_1, \dots, \lambda_k \in \mathbb{R}$:

$$(*) \qquad a^{(1)} = \lambda_1 v^{(1)} + \dots + \lambda_k v^{(k)}$$



Choose $\lambda_j \neq 0$:

$$v^{(j)} = \frac{1}{\lambda_j} \left(\lambda_1 v^{(1)} + \dots + \lambda_{j-1} v^{(j-1)} + \lambda_{j+1} v^{(j+1)} + \dots + \lambda_k v^{(k)} - a^{(1)} \right)$$

Remove $v^{(j)}$ from $\mathcal{B} \cup \mathcal{A}$ and call it $\mathcal{C}$.

$\mathcal{C}$ is linearly independent:

$$\tilde{\lambda}_1 v^{(1)} + \cdots + \tilde{\lambda}_{j-1} v^{(j-1)} + \tilde{\lambda}_j a^{(1)} + \tilde{\lambda}_{j+1} v^{(j+1)} + \cdots + \tilde{\lambda}_k v^{(k)} = 0$$

Assume $\tilde{\lambda}_j \neq 0$: $a^{(1)}$ = linear combination with $v^{(1)}, \dots, v^{(j-1)}, v^{(j+1)}, \dots, v^{(k)}$ $\lightning$ $(*)$

Hence: $\tilde{\lambda}_j = 0 \implies$

$$\tilde{\lambda}_1 v^{(1)} + \cdots + \tilde{\lambda}_{j-1} v^{(j-1)} + \tilde{\lambda}_{j+1} v^{(j+1)} + \cdots + \tilde{\lambda}_k v^{(k)} = 0$$

$$\overset{\text{lin. independence}}{\implies} \quad \tilde{\lambda}_i = 0 \quad \text{for } i \in \{1, \dots, k\}$$

$\mathcal{C}$ spans U : $u \in U \overset{\mathcal{B} \text{ basis}}{\implies}$ there are coefficients

$$\boxed{v^{(j)} = \frac{1}{\lambda_j}\left(\lambda_1 v^{(1)} + \cdots + \lambda_{j-1} v^{(j-1)} + \lambda_{j+1} v^{(j+1)} + \cdots + \lambda_k v^{(k)} - a^{(1)}\right)}$$

$$u = \mu_1 v^{(1)} + \cdots + \mu_{j-1} v^{(j-1)} + \mu_j v^{(j)} + \mu_{j+1} v^{(j+1)} + \cdots + \mu_k v^{(k)}$$

$$= \tilde{\mu}_1 v^{(1)} + \cdots + \tilde{\mu}_{j-1} v^{(j-1)} + \tilde{\mu}_j a^{(1)} + \tilde{\mu}_{j+1} v^{(j+1)} + \cdots + \tilde{\mu}_k v^{(k)}$$

Linear Algebra - Part 27

Steinitz Exchange Lemma: $(v^{(1)}, v^{(2)}, \dots, v^{(k)})$ basis of U

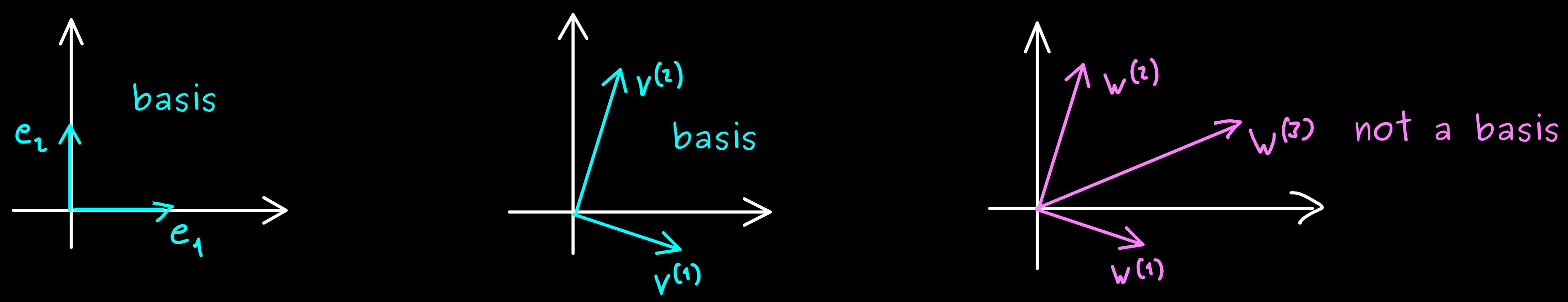
$(a^{(1)}, a^{(2)}, \dots, a^{(\ell)})$ lin. independent vectors in U

$\Longrightarrow$ new basis of U

Fact: Let $U \subseteq \mathbb{R}^n$ be a subspace and $\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ be a basis of U.

Then: (a) Each family $(w^{(1)}, w^{(2)}, \dots, w^{(m)})$ with $m > k$ vectors in U is linearly dependent.

(b) Each basis of U has exactly k elements.



Definition: Let $U \subseteq \mathbb{R}^n$ be a subspace and $\mathcal{B}$ be a basis of U.

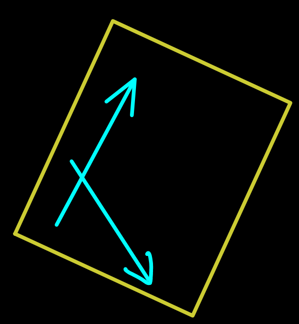
The number of vectors in $\mathcal{B}$ is called the dimension of U.

We write: $\dim(U)$ ← integer

Set: $\dim(\{0\}) := 0$ $\quad \left(\operatorname{Span}(\emptyset) = \{0\} \right)$ ← basis

Example: $(e_1, e_2, \dots, e_n)$ standard basis of $\mathbb{R}^n$

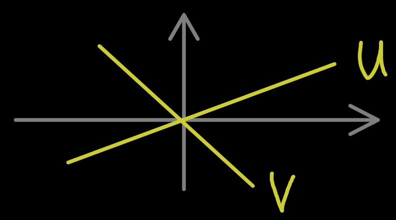
$$\dim(\mathbb{R}^n) = n$$



Linear Algebra – Part 28

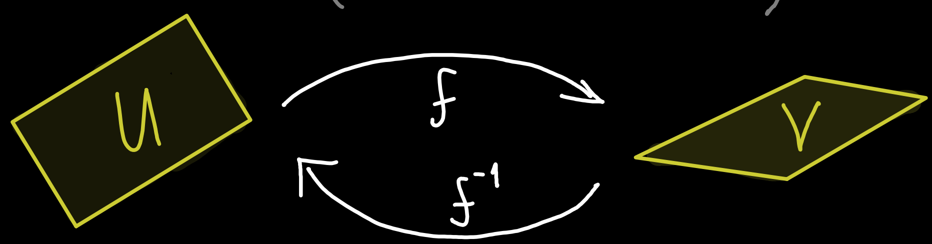
Dimension of U: number of elements in a basis of U $= \dim(U)$

Theorem: $U, V \subseteq \mathbb{R}^n$ linear subspaces



(a) $\dim(U) = \dim(V) \iff$ there is a bijective linear map $f: U \to V$

$\left(f^{-1}: V \to U \text{ linear} \right)$



(b) $U \subseteq V$ and $\dim(U) = \dim(V) \implies U = V$

Proof: (a) ($\Rightarrow$) We assume $\dim(U) = \dim(V)$.

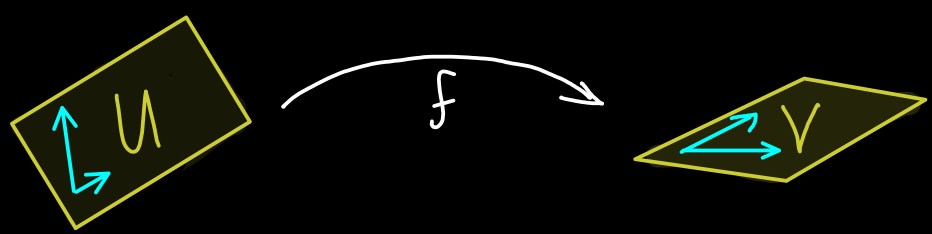
Hence:

$$\mathcal{B} = \left(u^{(1)}, u^{(2)}, \dots, u^{(k)} \right) \quad \text{basis of } U$$

$$\quad\;\; \downarrow \quad\;\; \downarrow \quad \dots \quad \downarrow$$

$$\mathcal{C} = \left(v^{(1)}, v^{(2)}, \dots, v^{(k)} \right) \quad \text{basis of } V$$

define: $f: U \to V$, $f(u^{(i)}) = v^{(i)}$



For $x \in U$: $f(x) = f\left(\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)} \right)$ uniquely determined $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

$$= \lambda_1 \cdot f(u^{(1)}) + \lambda_2 \cdot f(u^{(2)}) + \dots + \lambda_k \cdot f(u^{(k)})$$

$$= \lambda_1 \cdot v^{(1)} + \dots + \lambda_k \cdot v^{(k)} \quad =: f(x)$$

Now define: $f^{-1}: V \to U$, $f^{-1}(v^{(i)}) = u^{(i)}$

Then: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y \implies f$ is bijective+linear

$(\Leftarrow)$ We assume that there is $\underbrace{\text{bijective}}_{\text{injective+surjective}}$ linear map $f: U \to V$.

Let $\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ be a basis of U

$$\Longrightarrow \left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)}) \right) \quad \text{basis in } V\ ?$$

f injective $\swarrow$ $\searrow$ f surjective

linearly independent $\qquad$ $\text{span}\left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)})\right) = V$

$$\Longrightarrow \dim(U) = \dim(V)$$

(b) We show: $\boxed{U \subseteq V \text{ and } \dim(U) = \dim(V) \Longrightarrow U = V}$

$$(u^{(1)}, u^{(2)}, \dots, u^{(k)}) \text{ basis of } U \Longrightarrow (u^{(1)}, u^{(2)}, \dots, u^{(k)}) \text{ basis of } V$$

$$v = \underbrace{\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)}}_{\in U}$$

$$\Longrightarrow U = V$$

$\square$

Linear Algebra – Part 29

$A \in \mathbb{R}^{m \times n} \longleftrightarrow f_A : \mathbb{R}^n \to \mathbb{R}^m$ linear map

Definition: Identity matrix in $\mathbb{R}^{n \times n}$:

$$\mathbb{1}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

other notations: I_n, id, Id, E_n

Properties:

$\mathbb{1}_n B = B$ for $B \in \mathbb{R}^{n \times m}$

$A \cdot \mathbb{1}_n = A$ for $A \in \mathbb{R}^{m \times n}$

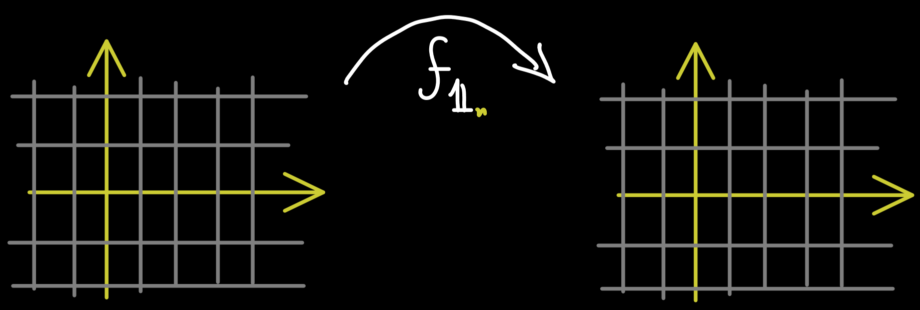
neutral element with respect to the matrix multiplication

Map level:

$f_{\mathbb{1}_n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x \longmapsto \underbrace{\mathbb{1}_n x}_{= x}$

$f_{\mathbb{1}_n}$ = identity map



Inverses: $A \in \mathbb{R}^{n \times n} \rightsquigarrow \tilde{A} \in \mathbb{R}^{n \times n}$ with $A\tilde{A} = \mathbb{1}$ and $\tilde{A}A = \mathbb{1}$

If such a $\tilde{A}$ exists, it's uniquely determined. Write A^{-1} (instead of $\tilde{A}$)

inverse of A

Definition: A matrix $A \in \mathbb{R}^{n \times n}$ is called invertible (= non-singular = regular)

if the corresponding linear map $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is bijective.

Otherwise we call A singular.

A matrix $\tilde{A} \in \mathbb{R}^{n \times n}$ is called the inverse of A if $f_{\tilde{A}} = (f_A)^{-1}$

Write A^{-1} (instead of $\tilde{A}$)

Summary:

$$\begin{aligned} f_{A^{-1}} \circ f_A &= id \\ f_A \circ f_{A^{-1}} &= id \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} A^{-1}A &= \mathbb{1} \\ AA^{-1} &= \mathbb{1} \end{aligned}$$

Linear Algebra - Part 30

injectivity, surjectivity, bijectivity for square matrices

system of linear equations: $Ax = b \overset{\text{if } A \text{ invertible}}{\Longrightarrow} A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$

Theorem: $A \in \mathbb{R}^{n \times n}$ square matrix. $f_A : \mathbb{R}^n \to \mathbb{R}^n$ induced linear map.

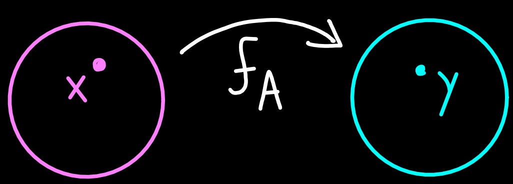
Then: f_A is injective $\Longleftrightarrow$ f_A is surjective

Proof: $(\Rightarrow)$ f_A injective, standard basis of $\mathbb{R}^n$ $(e_1, \dots, e_n)$

$$\Rightarrow \underbrace{(f_A(e_1), \dots, f_A(e_n))}_{\text{basis of } \mathbb{R}^n} \text{ still linearly independent}$$

$$\Rightarrow f_A \text{ is surjective}$$

$(\Leftarrow)$ f_A surjective



For each $y \in \mathbb{R}^n$, you find $x \in \mathbb{R}^n$ with $f_A(x) = y$.

We know: $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$$y = f_A(x) = x_1 f_A(e_1) + x_2 f_A(e_2) + \cdots + x_n f_A(e_n)$$

$$\Rightarrow (f_A(e_1), \dots, f_A(e_n)) \text{ spans } \mathbb{R}^n$$

$$\overset{n \text{ vectors}}{\Longrightarrow} (f_A(e_1), \dots, f_A(e_n)) \text{ linearly independent}$$

Assume $f_A(x) = f_A(\tilde{x}) \Rightarrow f_A(\underbrace{x - \tilde{x}}_{v}) = 0$

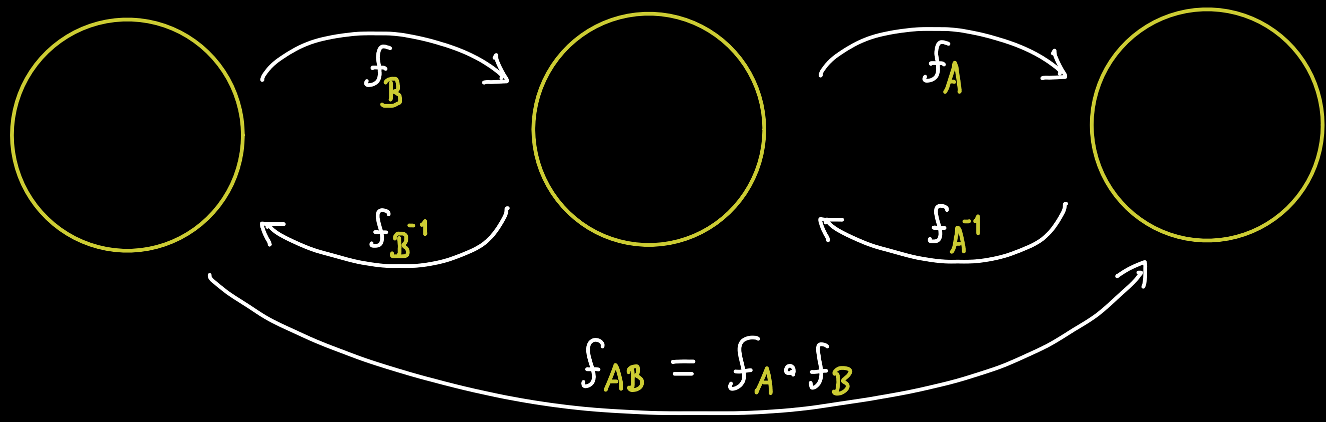
$$\Rightarrow v_1 f_A(e_1) + v_2 f_A(e_2) + \cdots + v_n f_A(e_n) = 0$$

$$\overset{\text{lin. independence}}{\Longrightarrow} v_1 = v_2 = \cdots = v_n = 0$$

$$\Rightarrow x = \tilde{x} \quad \Rightarrow \quad f_A \text{ is injective} \qquad \square$$

Linear Algebra - Part 31

matrices $A, B \in \mathbb{R}^{n \times n}$



We have: $f_{B^{-1}} \circ f_{A^{-1}} = (f_{AB})^{-1} \Rightarrow \boxed{(AB)^{-1} = B^{-1}A^{-1}}$

Important fact: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear and bijective

$\Rightarrow f^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is also linear

Proof:

$$f^{-1}(\lambda y) = f^{-1}(\lambda \cdot f(x)) \underbrace{=}_{f \text{ linear}} f^{-1}(f(\lambda x)) = \lambda \cdot x = \lambda f^{-1}(y) \checkmark$$

There is exactly one x with $f(x) = y$

$$f^{-1}(y + \tilde{y}) = f^{-1}(f(x) + f(\tilde{x})) \underbrace{=}_{f \text{ linear}} f^{-1}(f(x + \tilde{x})) = x + \tilde{x}$$

$$= f^{-1}(y) + f^{-1}(\tilde{y}) \checkmark$$

Linear Algebra - Part 32

Transposition: changing the roles of columns and rows

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

For $a \in \mathbb{R}^n$ we have: $(a^T)^T = a$

Definition: For $A \in \mathbb{R}^{m \times n}$ we define $A^T \in \mathbb{R}^{n \times m}$ (transpose of A) by:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Examples:

(a) $$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}$$

(b) $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

(c) $$A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}$$ (symmetric matrix)

Remember: $(AB)^T = B^T A^T$

Linear Algebra - Part 33

$$A \in \mathbb{R}^{m \times n} \rightsquigarrow A^T \in \mathbb{R}^{n \times m}$$

standard inner product in $\mathbb{R}^n \rightsquigarrow \langle u, v \rangle \in \mathbb{R}$, $= u^T v$

Proposition: For $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\underbrace{\langle y, Ax \rangle}_{\text{inner product in } \mathbb{R}^m} = \underbrace{\langle A^T y, x \rangle}_{\text{inner product in } \mathbb{R}^n}$$

Proof: $\langle \tilde{u}, \tilde{v} \rangle = \tilde{u}^T \tilde{v}$ for $\tilde{u}, \tilde{v} \in \mathbb{R}^m$

$$(A^T y)^T = y^T \cdot (A^T)^T$$

$$\langle y, \widetilde{Ax} \rangle = y^T (Ax) = (y^T A) x = (A^T y)^T x = \langle A^T y, x \rangle \qquad \square$$

Alternative definition: A^T is the only matrix $B \in \mathbb{R}^{n \times m}$ that satisfies:

$$\langle y, Ax \rangle = \langle By, x \rangle \quad \text{for all } x, y$$

Linear Algebra - Part 34

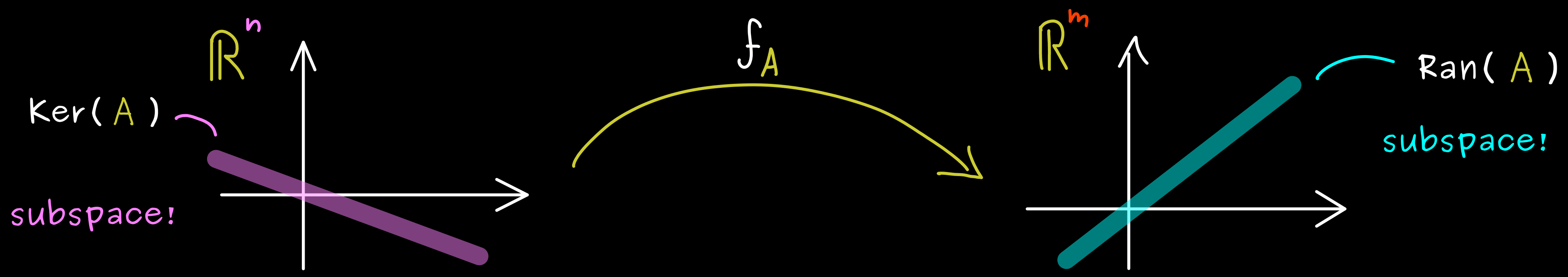
$A \in \mathbb{R}^{m \times n}$ induces a linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$

$$\mathrm{Ran}(A) := \{ Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$$ range of A (image of A)

$= \mathrm{Ran}(f_A)$ (see Start Learning Sets - Part 5)

$$\mathrm{Ker}(A) := \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subseteq \mathbb{R}^n$$ kernel of A (nullspace of A)

$= f_A^{-1}[\{0\}]$ preimage of $\{0\}$ under f_A



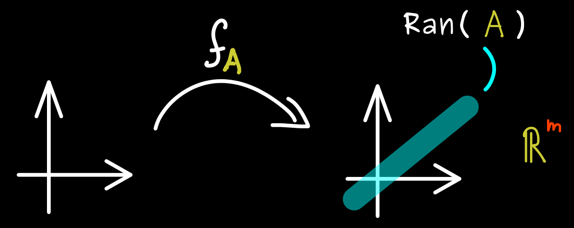
Remember: $\mathrm{Ran}(A) = \mathrm{Span}(a_1, a_2, \dots, a_n)$ $\quad A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$

Solving LES? $Ax = b$ existence of solutions: $b \in \mathrm{Ran}(A)$?

uniqueness of solutions: $\mathrm{Ker}(A) \neq \{0\}$?

Linear Algebra - Part 35

Definition: For $A \in \mathbb{R}^{m \times n}$ we define:



$$\text{rank}(A) := \dim(\text{Ran}(A))$$

$$= \dim(\text{Span of columns of } A)$$

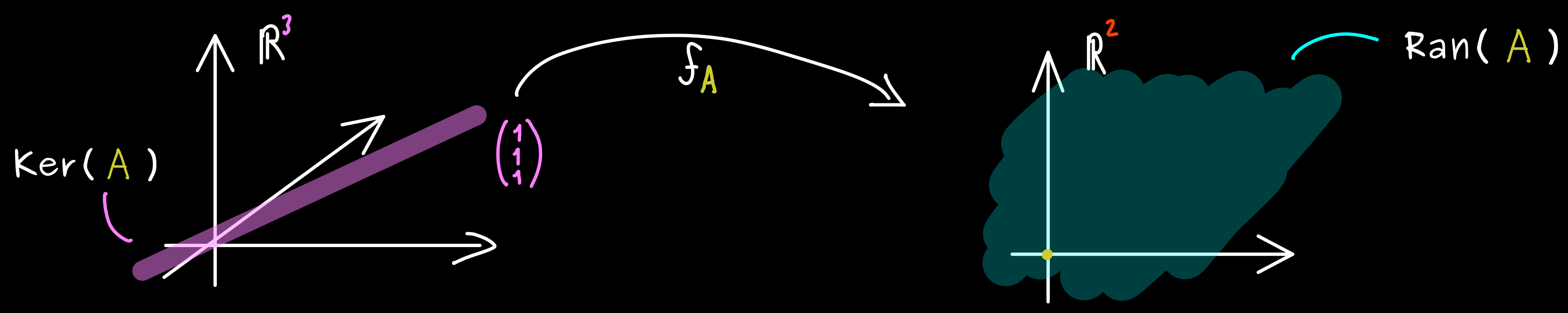
$$\leq \min(n, m)$$

A has <u>full rank</u> if $\text{rank}(A) = \min(n, m)$

Example: (a) $A = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}$, $\text{rank}(A) = 1$ (full rank)

(b) $A = \begin{pmatrix} 2 & 2 & -4 \\ 1 & 0 & -1 \end{pmatrix}$, $\text{rank}(A) = 2$ (full rank)

linearly independent



Definition: For $A \in \mathbb{R}^{m \times n}$ we define:

$$\text{nullity}(A) := \dim(\text{Ker}(A))$$

Rank-nullity theorem: For $A \in \mathbb{R}^{m \times n}$ (n columns)

$$\dim(\text{Ker}(A)) + \dim(\text{Ran}(A)) = n$$

<u>Proof:</u> $k = \dim(\mathrm{Ker}(A))$. Choose: $(b_1, \dots, b_k)$ basis of $\mathrm{Ker}(A)$.

Steinitz Exchange Lemma $\Longrightarrow$ $(b_1, \dots, b_k, c_1, \dots, c_r)$ basis of $\mathbb{R}^n$

$r := n - k$

$$\mathrm{Ran}(A) = \mathrm{Span}\big(\underbrace{Ab_1}_{=0}, \dots, \underbrace{Ab_k}_{=0}, Ac_1, \dots, Ac_r\big)$$

$$= \mathrm{Span}\big(Ac_1, \dots, Ac_r\big) \quad \Longrightarrow \quad \dim(\mathrm{Ran}(A)) \leq r$$

<u>To show:</u> $(Ac_1, \dots, Ac_r)$ is linearly independent

$$\lambda_1 Ac_1 + \lambda_2 Ac_2 + \cdots + \lambda_r Ac_r = 0$$

$$\overset{\text{linearity}}{=} A\Big(\sum_{i=1}^{r} \lambda_i c_i\Big) \quad \Longrightarrow \quad \sum_{i=1}^{r} \lambda_i c_i \in \mathrm{Ker}(A)$$

$$\overset{\text{basis of kernel}}{\Longrightarrow} \quad \sum_{i=1}^{r} \lambda_i c_i = \sum_{j=1}^{k} \mu_j b_j \quad \Longrightarrow \quad \sum_{i=1}^{r} \lambda_i c_i + \sum_{j=1}^{k} (-\mu_j) b_j = 0$$

$$\Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$$

$$\Longrightarrow \quad \dim(\mathrm{Ran}(A)) = r \qquad \square$$

Linear Algebra - Part 36

System of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 1 \\ 4x_1 + 6x_2 + 9x_3 &= 1 \\ 2x_1 + 4x_2 + 6x_3 &= 1 \end{aligned}$$

3 equations
3 unknowns

Short notation: $Ax = b \xrightarrow{\text{augmented matrix}} (A \mid b)$

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 6 & 9 & 1 \\ 2 & 4 & 6 & 1 \end{array}\right)$$

Example:

$x_1 + 3x_2 = 7$ (equation 1)

$2x_1 - x_2 = 0$ (equation 2) $\rightsquigarrow$ $x_2 = 2x_1$

put in equation 1

$\Rightarrow x_1 + 3(2x_1) = 7$

$\Leftrightarrow 7x_1 = 7 \Leftrightarrow x_1 = 1 \rightsquigarrow x_2 = 2$

$\Rightarrow$ Only possible solution: $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Check? ✓

$\Rightarrow$ The system has a unique solution given by $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Better method: Gaussian elimination

Example:

$x_1 + 3x_2 = 7$ (equation 1)

$2x_1 - x_2 = 0$ (equation 2) $- 2\cdot$(equation 1)

eliminate x_1

$\rightsquigarrow$

$x_1 + 3x_2 = 7$ (equation 1)

$0 - 7x_2 = -14$ (equation 2)' $\cdot \left(-\frac{1}{7}\right)$

$\rightsquigarrow$

$x_1 + 3x_2 = 7$ (equation 1)

$x_2 = 2$ (equation 2)''

$\Rightarrow x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ solution

Linear Algebra – Part 37

$$Ax = b \quad \overset{\text{augmented matrix}}{\rightsquigarrow} \quad (A \mid b)$$

$$A \overset{\text{reversible manipulation}}{\leftrightsquigarrow} \tilde{A} \; : \qquad \underbrace{M}_{\text{invertible}} A = \tilde{A} \quad \leftrightsquigarrow \quad A = M^{-1}\tilde{A}$$

For the system of linear equations:

$$Ax = b \quad \leftrightsquigarrow \quad MAx = Mb \qquad \text{(new system)}$$

Example:

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \rightsquigarrow MA = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{———} & \alpha_1^T & \text{———} \\ & \vdots & \\ \text{———} & \alpha_m^T & \text{———} \end{pmatrix}$$

$$c^T = (0, \dots, 0, c_i, 0, \dots, 0, c_j, 0, \dots, 0) \quad \Rightarrow \quad c^T A = c_i \, \alpha_i^T + c_j \, \alpha_j^T$$

Example:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} \text{———} & \alpha_1^T & \text{———} \\ \text{———} & \alpha_2^T & \text{———} \\ \text{———} & \alpha_3^T & \text{———} \end{pmatrix} = \begin{pmatrix} \text{———} & \alpha_1^T & \text{———} \\ \text{———} & \alpha_2^T & \text{———} \\ & \alpha_3^T + \lambda \cdot \alpha_1^T & \end{pmatrix}$$

invertible with inverse: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$

Definition: $Z_{i+\lambda j} \in \mathbb{R}^{m \times m}$, $i \neq j$, $\lambda \in \mathbb{R}$,

defined as the identity matrix with λ at the (i,j)th position.

Example: (exchanging rows)

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_{1\leftrightarrow 3}} \begin{pmatrix} — & \alpha_1^T & — \\ — & \alpha_2^T & — \\ — & \alpha_3^T & — \end{pmatrix} = \begin{pmatrix} — & \alpha_3^T & — \\ — & \alpha_2^T & — \\ — & \alpha_1^T & — \end{pmatrix}$$

Definition: $P_{i\leftrightarrow j} \in \mathbb{R}^{m\times m}$, $i \neq j$, defined as the identity matrix where the ith and the jth rows are exchanged.

Definition: (scaling rows)

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} — & \alpha_1^T & — \\ & \vdots & \\ — & \alpha_m^T & — \end{pmatrix} = \begin{pmatrix} — & d_1\,\alpha_1^T & — \\ & \vdots & \\ — & d_m\,\alpha_m^T & — \end{pmatrix}$$

with $d_k \neq 0$

Definition: row operations: finite combination of $Z_{i+\lambda j}$, $P_{i\leftrightarrow j}$, $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}$, ...

(for example: $M = Z_{3+7\,1}\, Z_{2+8\,1}\, P_{1\leftrightarrow 2}$)

Property: For $A \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{R}^{m\times m}$ (invertible), we have:

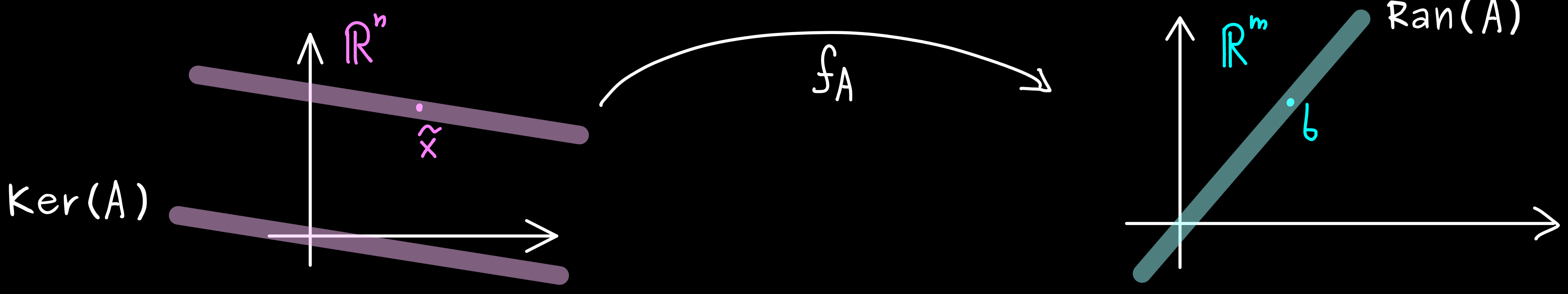
$$\mathrm{Ker}(MA) = \mathrm{Ker}(A), \quad \mathrm{Ran}(MA) = M\,\mathrm{Ran}(A)$$

$$:= \{My \mid y \in \mathrm{Ran}(A)\}$$

Linear Algebra – Part 38

Set of solutions: $Ax = b$ $\quad (A \in \mathbb{R}^{m \times n})$

solution: $\tilde{x}$ satisfies $A\tilde{x} = b$



uniqueness needs $\mathrm{Ker}(A) = \{0\}$

existence needs $b \in \mathrm{Ran}(A)$

Proposition: For a system $Ax = b$ $\quad (A \in \mathbb{R}^{m \times n})$

the set of solutions $\quad S := \{ \tilde{x} \in \mathbb{R}^n \mid A\tilde{x} = b \}$

is an affine subspace (or empty).

More concretely: We have either $S = \emptyset$

or $S = v_0 + \mathrm{Ker}(A)$ $\quad$ for a vector $v_0 \in \mathbb{R}^n$

$$v_0 + \mathrm{Ker}(A) = \{ v_0 + x_0 \mid x_0 \in \mathrm{Ker}(A) \}$$

Proof: Assume $v_0 \in S$. $\Rightarrow$ $Av_0 = b$

Set $\tilde{x} := v_0 + x_0$ for a vector $x_0 \in \mathbb{R}^n$.

Then: $\tilde{x} \in S \Leftrightarrow A\underbrace{\tilde{x}}_{(v_0 + x_0)} = b \Leftrightarrow \underbrace{Av_0}_{= b} + Ax_0 = b$

$$\Leftrightarrow Ax_0 = 0 \Leftrightarrow x_0 \in \mathrm{Ker}(A) \qquad \square$$

Remember: Row operations don't change the set of solutions!

$$S = v_0 + \underbrace{\mathrm{Ker}(A)}_{= \mathrm{Ker}(MA)}$$

$Av_0 = b$

$\Leftrightarrow MAv_0 = Mb$

$\rightsquigarrow$ Gaussian elimination
- decide $b \in \mathrm{Ran}(A)$
- gives us a particular solution v_0
- gives us $\mathrm{Ker}(A)$

Linear Algebra - Part 39

Goal: Gaussian elimination (named after Carl Friedrich Gauß)

solve $Ax = b$

↳ use row operations to bring $(A|b)$ into upper triangular form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

backwards substitution:

third row: $3x_3 = 1 \Rightarrow \boxed{x_3 = \frac{1}{3}}$

second row: $2x_2 + x_3 = 1 \Rightarrow \boxed{x_2 = \frac{1}{3}}$

first row: $1x_1 + 2x_2 + 3x_3 = 1 \Rightarrow x_1 = -\frac{2}{3}$

↳ or use row operations to bring $(A|b)$ into row echelon form

↳ construct solution set

Example: system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 - 1x_3 &= 4 \\ 2x_1 - 1x_2 + 7x_3 &= 0 \\ 6x_1 + 13x_2 - 4x_3 &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 2 & -1 & 7 & 0 \\ 6 & 13 & -4 & 9 \end{array}\right) \begin{array}{l} \\ -1\cdot \mathrm{I} \\ -3\cdot \mathrm{I} \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 4 & -1 & -3 \end{array}\right) \begin{array}{l} \\ \\ +1\cdot \mathrm{II} \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 7 & -7 \end{array}\right) \xrightarrow[\text{substitution}]{\text{backwards}} \begin{aligned} x_3 &= -1 \\ x_2 &= -1 \\ x_1 &= 3 \end{aligned}$$

set of solutions: $S = \left\{ \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}$

Gaussian elimination:

$$\left(\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \cdots & a_{1n} & b_1 \\ \boxed{a_{21}} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \boxed{a_{m1}} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} - & \alpha_1^T & - \\ - & \alpha_2^T & - \\ & \vdots & \\ - & \alpha_m^T & - \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \alpha_1^T \\ \alpha_2^T - \frac{a_{21}}{a_{11}} \alpha_1^T \\ \vdots \\ \alpha_m^T - \frac{a_{m1}}{a_{11}} \alpha_1^T \end{pmatrix} \xrightsquigarrow{\text{continue iteratively}} \cdots$$

row echelon form

Linear Algebra – Part 40

Row echelon form

$$A = \begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & \boxed{4} & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: A matrix $A \in \mathbb{R}^{m \times n}$ is in row echelon form if:

(1) All zero rows (if there are any) are at the bottom.

(2) For each row: the first non-zero entry is strictly to the right of the first non-zero entry of the row above.

pivots

$$A = \begin{pmatrix} \boxed{1} & 3 & 5 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} \boxed{1} & 3 & 5 & 0 & 1 \\ 0 & \boxed{2} & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

variables with no pivot in their columns are called **free variables** (x_3)

variables with a pivot in their columns are called **leading variables** (x_1, x_2, x_4)

Procedure: $Ax = b \rightsquigarrow (A \mid b) \xrightarrow[\text{row operations}]{\text{Gaussian elimination}} (A' \mid b')$ row echelon form

$\Downarrow$

solutions S $\leftarrow$ backwards substitution $\leftarrow$ put free variable to the right-hand side

Example:

free variables

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left(\begin{array}{ccccc|c}
\boxed{1} & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & \boxed{2} & -1 & 4 & 2 \\
0 & 0 & 0 & \boxed{4} & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

$$
\rightsquigarrow
\begin{array}{c}
\begin{array}{ccc} x_1 & x_3 & x_4 \end{array} \\
\left(\begin{array}{ccc|c}
\boxed{1} & 0 & 1 & 3 - 2x_2 \\
0 & \boxed{2} & -1 & 2 - 4x_5 \\
0 & 0 & \boxed{4} & 8 - 8x_5 \\
0 & 0 & 0 & 0
\end{array}\right)
\end{array}
\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \end{array}
$$

III $\quad 4x_4 = 8 - 8x_5 \quad \Rightarrow \quad \boxed{x_4 = 2 - 2x_5} \qquad x_5 \in \mathbb{R}$

II $\quad 2x_3 - x_4 = 2 - 4x_5$

$\Rightarrow \quad 2x_3 - 2 + 2x_5 = 2 - 4x_5 \quad \Rightarrow \quad 2x_3 = 4 - 6x_5 \quad \Rightarrow \quad \boxed{x_3 = 2 - 3x_5}$

I $\quad x_1 + x_4 = 3 - 2x_2 \quad \Rightarrow \quad x_1 + 2 - 2x_5 = 3 - 2x_2 \quad \Rightarrow \quad \boxed{x_1 = 1 - 2x_2 + 2x_5}$

set of solutions:

$$
S = \left\{ \begin{pmatrix} 1 - 2x_2 + 2x_5 \\ x_2 \\ 2 - 3x_5 \\ 2 - 2x_5 \\ x_5 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}
$$

$$
= \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}
$$

Linear Algebra - Part 41

$A \in \mathbb{R}^{m \times n} \xrightarrow{\text{Gaussian elimination}}$ row echelon form

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 & 0 \\ 0 & 0 & 0 & \boxed{4} & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \quad \mathrm{Ker}(A) = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$

Remember:

$\dim(\mathrm{Ker}(A))$ = number of free variables

$+$

$\dim(\mathrm{Ran}(A))$ = number of leading variables

$= n$

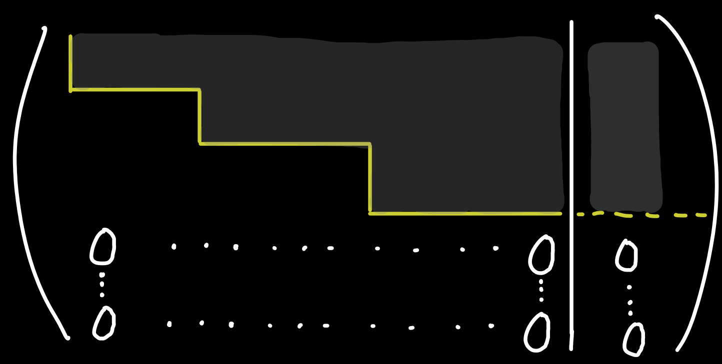
Proposition: For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we have the following equivalences:

(1) $Ax = b$ has at least one solution.

(2) $b \in \mathrm{Ran}(A)$

(3) b can be written as a linear combination of the columns of A.

(4) Row echelon form looks like:



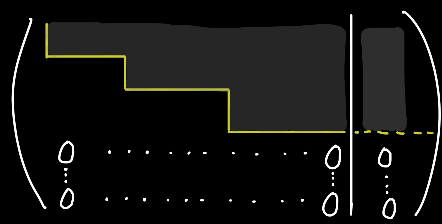
Proof: (1) $\Leftrightarrow$ (2) given by definition of $\text{Ran}(A)$

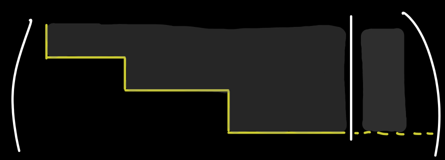
(2) $\Leftrightarrow$ (3) given by column picture of $\text{Ran}(A)$

$$\text{Ran}(A) = \left\{ \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} x \;\middle|\; x \in \mathbb{R}^n \right\}$$

$$= \left\{ x_1 \cdot \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$$

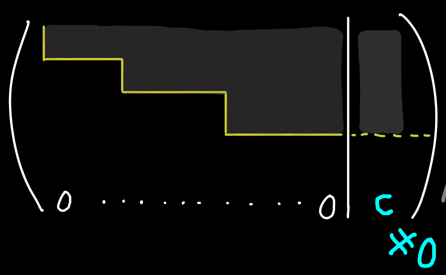
(4) $\Rightarrow$ (1)

Assume we have this: 

Then solve  by backwards substitution.

(or argue with $\text{rank}(A) = \text{rank}((A|b))$)

(1) $\Rightarrow$ (4) (let's show: $\neg(4) \Rightarrow \neg(1)$)

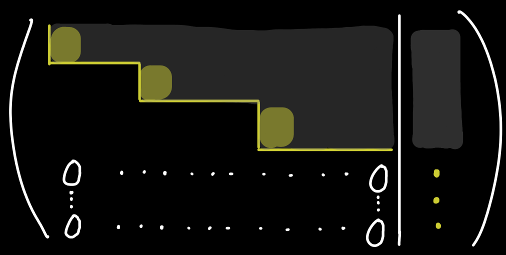
Assume: 

not solvable $0 = c$ ↯

$\Rightarrow$ no solution for $Ax = b$ $\square$

Linear Algebra - Part 42

$Ax = b \rightsquigarrow$ row echelon form



$S = \emptyset$ or $S = v_0 + \mathrm{Ker}(A)$

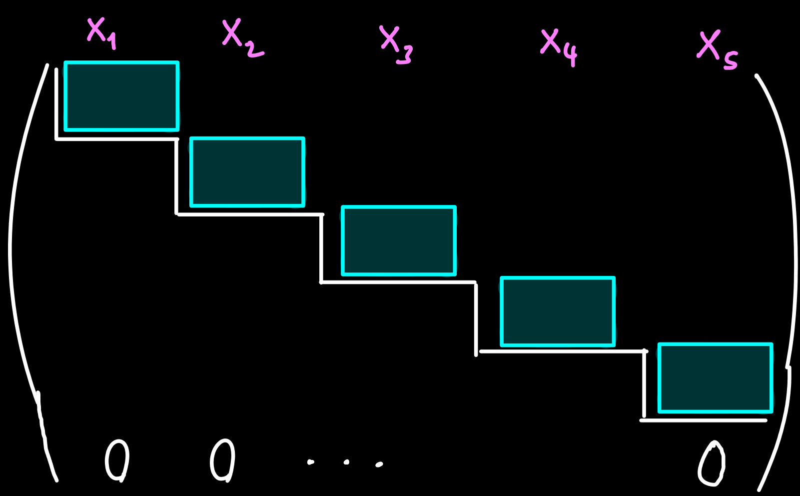
Proposition: For $A \in \mathbb{R}^{m \times n}$, we have the following equivalences:

(a) For every $b \in \mathbb{R}^m$: $Ax = b$ has at most one solution.

(b) $\mathrm{Ker}(A) = \{0\}$

(c) Row echelon form looks like:

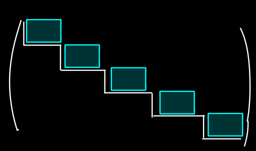
$\boxed{\text{every column has a pivot}}$



(d) $\mathrm{rank}(A) = n$

(e) The linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ is injective.

Result for square matrices: For $A \in \mathbb{R}^{n \times n}$:



$\mathrm{Ker}(A) = \{0\} \iff \mathrm{Ran}(A) = \mathbb{R}^n \iff Ax = b$ has a unique solution for some $b \in \mathbb{R}^n$

$\Updownarrow \qquad\qquad \Updownarrow$

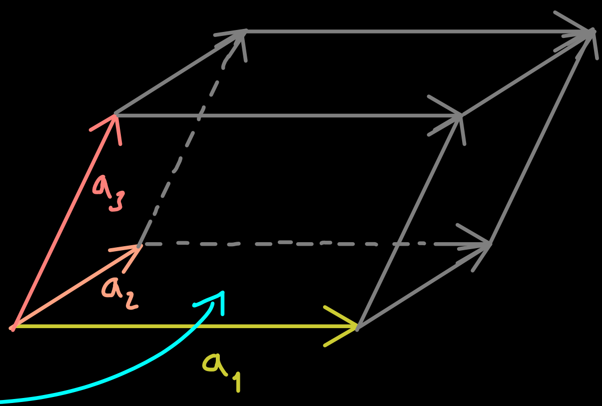
f_A injective $\iff$ f_A surjective $\iff$ $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$

Linear Algebra - Part 43

$A \in \mathbb{R}^{n \times n} \rightsquigarrow \det(A) \in \mathbb{R}$ with properties:

(1) $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$, columns span a parallelepiped

volume $= |\det(A)|$



(2) $\det(A) = 0 \iff \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix}, \dots, \begin{pmatrix} | \\ a_n \\ | \end{pmatrix}$ linearly dependent

$\iff A$ is not invertible

(3) sign of $\det(A)$ gives orientation $\left(\det(\mathbb{1}_n) = +1 \right)$

Linear Algebra - Part 44

$A \in \mathbb{R}^{2\times 2} \rightsquigarrow$ system of linear equations $Ax = b$

Assume $a_{11} \neq 0$

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) \xrightarrow{II - \frac{a_{21}}{a_{11}} I} \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array}\right) \xrightarrow{II \cdot a_{11}}$$

$$\rightsquigarrow \left(\begin{array}{cc|c} \boxed{a_{11}} & a_{12} & b_1 \\ 0 & \boxed{a_{11}a_{22} - a_{21}a_{12}} & a_{11} \cdot b_2 - a_{21} b_1 \end{array}\right)$$

$\neq 0 \iff$ we have a unique solution

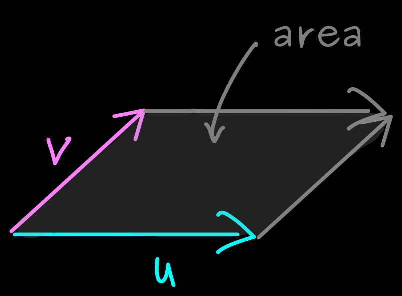
Definition: For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, the number

$$\det(A) := a_{11}a_{22} - a_{12}a_{21}$$

is called the determinant of A.

What about volumes? $\rightsquigarrow$ vol_n

in $\mathbb{R}^2$: $\mathrm{vol}_2(u, v) :=$ orientated area of parallelogram



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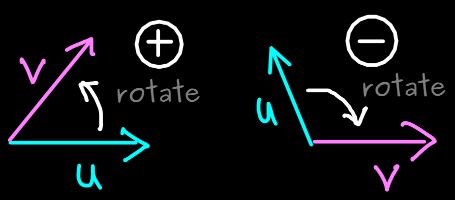
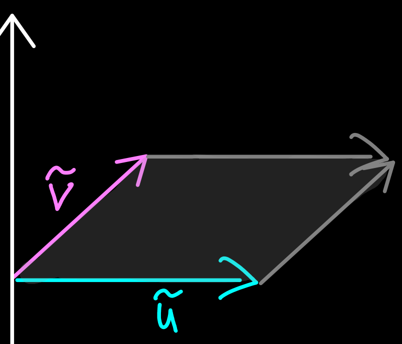


Relation to cross product: embed $\mathbb{R}^2$ into $\mathbb{R}^3$: $\tilde{u} := \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$, $\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

$\mathbb{R}^3$:



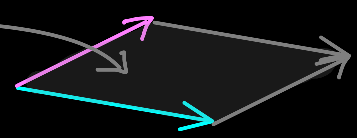
$$\| \tilde{u} \times \tilde{v} \| = \left\| \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \right\| = |\underbrace{u_1 v_2 - v_1 u_2}_{\det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}}|$$

Result: $\mathrm{vol}_2(u, v) = \det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$ (volume function = determinant)

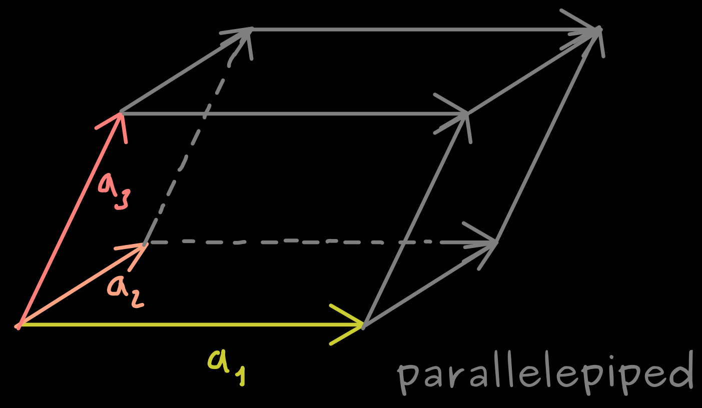
Linear Algebra - Part 45

volume measure?

- area in $\mathbb{R}^2$

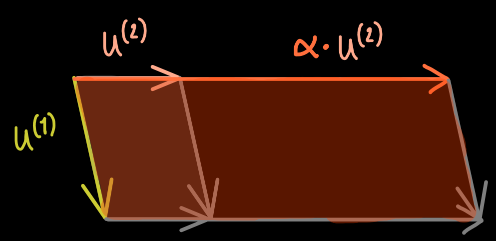


- n-dimensional volume $\mathbb{R}^n$



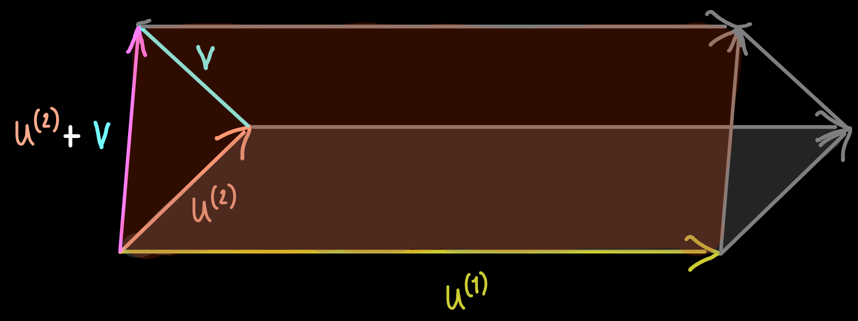
Definition: $\mathrm{vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$ is called n-dimensional volume function if

(a) $\mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, \alpha \cdot u^{(j)}, \dots, u^{(n)}) = \alpha \cdot \mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(n)})$



for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$
for all $\alpha \in \mathbb{R}$
for all $j \in \{1, \dots, n\}$

(b) $\mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)} + v, \dots, u^{(n)}) = \mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(n)}) + \mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, v, \dots, u^{(n)})$



for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$
for all $v \in \mathbb{R}^n$
for all $j \in \{1, \dots, n\}$

(c) $\mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(i)}, \dots, u^{(j)}, \dots, u^{(n)})$

$= - \mathrm{vol}_n(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(i)}, \dots, u^{(n)})$

for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$
for all $i, j \in \{1, \dots, n\}$
$i \neq j$

(d) $\mathrm{vol}_n(e_1, e_2, \dots, e_n) = 1$ (unit hypercube)

Result in $\mathbb{R}^2$:

$$\mathrm{vol}_2\left(\begin{pmatrix}a\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) = \mathrm{vol}_2\left(\begin{pmatrix}a\\0\end{pmatrix}+\begin{pmatrix}0\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right)$$

$$\overset{(b)}{=} \mathrm{vol}_2\left(\begin{pmatrix}a\\0\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) + \mathrm{vol}_2\left(\begin{pmatrix}0\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right)$$

$$\overset{(a)}{=} a\cdot\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) + c\cdot\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right)$$

$$\overset{(b)}{=} a\cdot\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}b\\0\end{pmatrix}\right) + a\cdot\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\d\end{pmatrix}\right) + c\cdot\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}b\\0\end{pmatrix}\right) + c\cdot\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\d\end{pmatrix}\right)$$

$$\overset{(b)}{=} a\cdot b\,\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)}_{=0} + a\cdot d\,\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)}_{\overset{(d)}{=}1} + c\cdot b\,\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)}_{=-1} + c\cdot d\cdot\underbrace{\mathrm{vol}_2\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)}_{=0}$$

$$\overset{(c),(d)}{=} a\cdot d - b\cdot c = \det\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

Define: $$\det\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} = \mathrm{vol}_n\left(\begin{pmatrix}a_{11}\\ \vdots\\ a_{n1}\end{pmatrix},\begin{pmatrix}a_{12}\\ \vdots\\ a_{n2}\end{pmatrix},\dots,\begin{pmatrix}a_{1n}\\ \vdots\\ a_{nn}\end{pmatrix}\right)$$

Linear Algebra – Part 46

n-dimensional volume form: $\mathrm{vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$

- linear in each entry
- antisymmetric
- $\mathrm{vol}_n(e_1, e_2, \ldots, e_n) = 1$

Let's calculate:

$$\mathrm{vol}_n\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \underbrace{\begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}}_{(*)} \right) = \mathrm{vol}_n\left(a_{11} \cdot e_1 + \cdots + a_{n1} e_n, (*) \right)$$

$$= a_{11} \cdot \mathrm{vol}_n(e_1, (*)) + \cdots + a_{n1} \cdot \mathrm{vol}_n(e_n, (*))$$

$$= \sum_{j_1=1}^{n} a_{j_1,1} \, \mathrm{vol}_n(e_{j_1}, (*)) = \sum_{j_1=1}^{n} a_{j_1,1} \, \mathrm{vol}_n\left(e_{j_1}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

$$= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} a_{j_1,1} \, a_{j_2,2} \cdot \mathrm{vol}_n\left(e_{j_1}, e_{j_2}, \begin{pmatrix} a_{13} \\ \vdots \\ a_{n3} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right)$$

$$= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{j_1,1} \, a_{j_2,2} \cdots a_{j_n,n} \cdot \underbrace{\mathrm{vol}_n(e_{j_1}, e_{j_2}, \ldots, e_{j_n})}_{= 0 \text{ if two indices coincide}}$$

permutation of $\{1, \ldots, n\}$

$$= \sum_{(j_1, \ldots, j_n) \in S_n} a_{j_1,1} \, a_{j_2,2} \cdots a_{j_n,n} \cdot \underbrace{\mathrm{vol}_n(e_{j_1}, e_{j_2}, \ldots, e_{j_n})}_{= \{1, -1\}}$$

where all entries are different

set of all permutations of $\{1, \ldots, n\}$

$$\mathrm{sgn}((j_1, \ldots, j_n)) = \begin{cases} +1, & \text{even number of exchanges to get to } (1, \ldots, n) \\ -1, & \text{odd number of exchanges to get to } (1, \ldots, n) \end{cases}$$

$$= \sum_{(j_1, \ldots, j_n) \in S_n} \mathrm{sgn}((j_1, \ldots, j_n)) \, a_{j_1,1} \, a_{j_2,2} \cdots a_{j_n,n} = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

(Leibniz formula)

Linear Algebra – Part 47

Leibniz formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} = \sum_{(j_1,\dots,j_n)\in S_n} \operatorname{sgn}((j_1,\dots,j_n))\, a_{j_1,1}\, a_{j_2,2} \cdots a_{j_n,n}$$

how many terms?

For $n=2$: $(1,2)$, $(2,1)$ — 2 permutations

For $n=3$: $(1,2,3), (2,3,1), (3,1,2)$, $(1,3,2), (3,2,1), (2,1,3)$ — 6 permutations (rule of Sarrus)

For $n=4$: … — 24 permutations

For n: … — $n!$ permutations

Rule of Sarrus:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = + \quad + \quad + \quad - \quad - \quad -$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

Example:

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 4 & 1\end{pmatrix} = \underline{-1} + 8 + \underline{(-4)} \; \underline{-(-1)} \; \underline{-(-8)} \; \underline{-4} = 8$$

Linear Algebra - Part 48

4x4-matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11}\cdot\det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$

24 permutations

6 permutations

$$- a_{21}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} & \cancel{a_{24}} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

6 permutations

$$+ a_{31}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} & \cancel{a_{34}} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

6 permutations

$$- a_{41}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \cancel{a_{41}} & \cancel{a_{42}} & \cancel{a_{43}} & \cancel{a_{44}} \end{pmatrix}$$

6 permutations

checkerboard

$$\begin{pmatrix} + & - & + & \\ - & + & - & \\ + & - & & \ddots \\ - & + & & \end{pmatrix}$$

Idea: $n\times n \rightsquigarrow (n-1)\times(n-1) \rightsquigarrow \cdots \rightsquigarrow 3\times 3 \rightsquigarrow 2\times 2 \rightsquigarrow 1\times 1$

Laplace expansion: $A\in\mathbb{R}^{n\times n}$. For jth column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij}\cdot\det\left(A^{(i,j)}\right)$$ expanding along the jth column

ith row and jth column are deleted

For ith row:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\cdot\det\left(A^{(i,j)}\right)$$ expanding along the ith row

Example:

expaning along 2nd row

$$\det\begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix} = -2\cdot\det\begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= (-2)(-1)\,1\cdot\det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2\cdot(6-4) = 4$$

Linear Algebra - Part 49

Triangular matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ & a_{22} & \ddots & & \vdots \\ & & a_{33} & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & a_{nn} \end{pmatrix} = a_{11} \cdot a_{22} \cdots a_{nn}$$

Block matrices:

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} & b_{11} & b_{12} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ a_{m1} & \cdots & a_{mm} & b_{m1} & & \cdots & b_{mk} \\ 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots & c_{21} & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & c_{k1} & & \cdots & c_{kk} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

$$\Longrightarrow \quad \det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \cdot \det(C)$$

Proposition: $\det(A^T) = \det(A)$

Proposition: $A, B \in \mathbb{R}^{n \times n}$: $\boxed{\det(A \cdot B) = \det(A) \cdot \det(B)}$ multiplicative map

If A is invertible, then: $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$$\det(A^{-1} B A) = \det(B)$$

Linear Algebra - Part 50

determinant is multiplicative: $\det(MA) = \det(M)\cdot\det(A)$

Gaussian elimination: $A \underset{\text{row operations}}{\rightsquigarrow} MA$ (see part 37)

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ — \alpha_3^T — \end{pmatrix} = \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ \alpha_3^T + \lambda \cdot \alpha_1^T \end{pmatrix}$$

$$Z_{3+\lambda 1} \Rightarrow \det(Z_{3+\lambda 1}) = 1$$

Adding rows with $Z_{i+\lambda j}$ $(i \neq j,\ \lambda \in \mathbb{R})$ does not change the determinant!

Exchanging rows with $P_{i \leftrightarrow j}$ $(i \neq j)$ does change the sign of the determinant!

Scaling one row with factor d_j scales the determinant by d_j!

Column operations? $\det(A^T) = \det(A)$ ✓

Example:

$$\det\begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 4 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} \overset{\substack{\text{rows} \\ \text{II} - 1\cdot\text{V}}}{=} \det\begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix}$$

$$\overset{\text{Laplace expansion}}{=} (+1)\cdot \det\begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

$$\overset{\substack{\text{columns} \\ \text{II} - 2\,\text{IV} \\ \text{III} + \text{IV}}}{=} \det\begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix}$$

$$\overset{\text{Laplace expansion}}{=} (+2)\cdot \det\begin{pmatrix} -1 & 1 & -2 \\ 1 & -2 & -2 \\ 1 & -4 & 3 \end{pmatrix} = 2\cdot 13 = 26$$

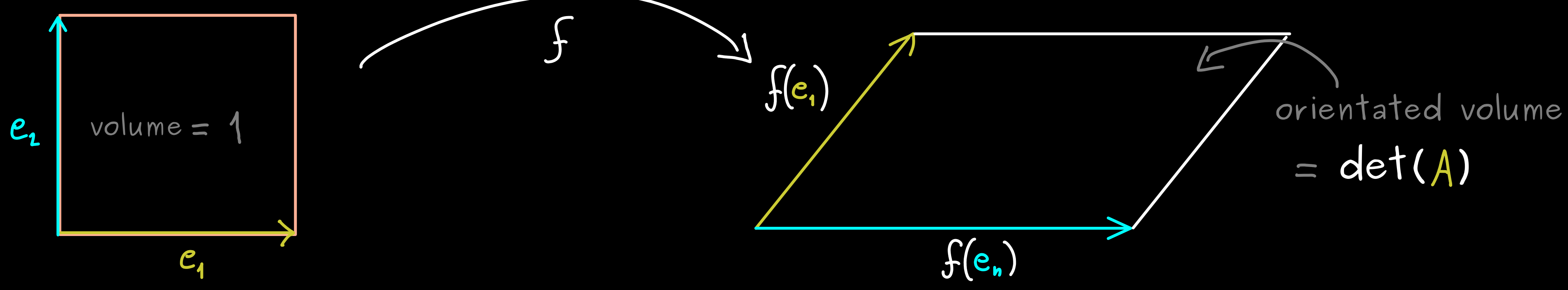
Linear Algebra - Part 51

matrix $A \in \mathbb{R}^{n \times n}$ $\rightsquigarrow$ linear map $f_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$

linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ $\rightsquigarrow$ there is exactly one $A \in \mathbb{R}^{n \times n}$ with $f = f_A$

Here: $A = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}$

unit cube in $\mathbb{R}^n$



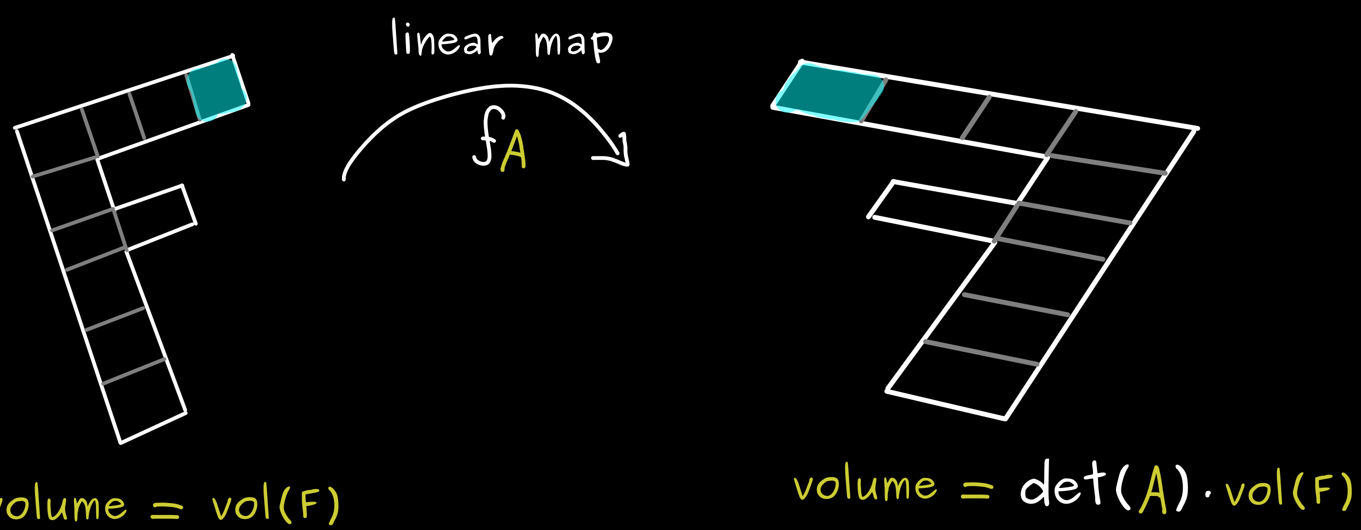
Remember: $\det(A)$ gives the relative change of volume caused by f_A.

Definition: For a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$, we define the determinant:

$$\det(f) := \det(A) \quad \text{where } A \text{ is } \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}$$

Multiplication rule: $\det(f \circ g) = \det(f)\det(g)$

Volume change:



Linear Algebra – Part 52

We know for $A \in \mathbb{R}^{2 \times 2}$: $\det(A) \neq 0 \iff Ax = b$ has a unique solution

$\iff A$ invertible = non-singular

For $A \in \mathbb{R}^{n \times n}$: $\det(A) = 0 \iff A$ singular

Proposition: For $A \in \mathbb{R}^{n \times n}$, the following claims are equivalent:

- $\det(A) \neq 0$
- columns of A are linearly independent
- rows of A are linearly independent
- $\text{rank}(A) = n$
- $\text{Ker}(A) = \{0\}$
- A is invertible
- $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$

Cramer's rule: $A \in \mathbb{R}^{n \times n}$ non-singular, $b \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ unique solution of $Ax = b$.

Then:

$$x_i = \frac{\det \begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & b & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}}{\det \begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & a_i & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}}$$

Proof: Use cofactor matrix $C \in \mathbb{R}^{n\times n}$ defined: $c_{ij} = (-1)^{i+j} \cdot \det\left(\cancel{A}\right)$ jth column deleted, ith row deleted

$$\overset{\text{Laplace expansion}}{=} \det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{j-1} & e_i & a_{j+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}$$

We can show: $$A^{-1} = \frac{C^T}{\det(A)}$$

Hence: $$x = A^{-1} b = \frac{C^T b}{\det(A)}$$ and $$(C^T b)_i = \sum_{k=1}^{n} (C^T)_{ik} b_k = \sum_{k=1}^{n} c_{ki} b_k$$

$$= \sum_{k=1}^{n} \det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & e_k & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix} b_k$$

$$\overset{\text{linear in the } i\text{th column}}{=} \det\begin{pmatrix} | & & | & b_1 & | & & | \\ a_1 & \cdots & a_{i-1} & b_2 & a_{i+1} & \cdots & a_n \\ & & & \vdots & & & \\ | & & | & b_n & | & & | \end{pmatrix}$$

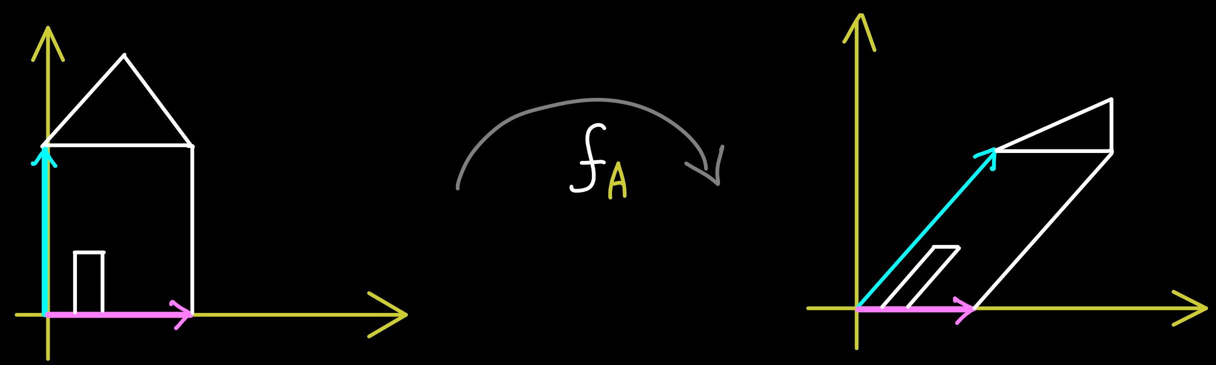
$\square$

Linear Algebra - Part 53

eigenvalue (German: Eigenwert) (David Hilbert, 1904)

↳ proper/own/characteristic

Consider: $A \in \mathbb{R}^{n \times n} \leftrightsquigarrow f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear map



Question: Are there vectors which are only scaled by f_A?

Answer: $Ax = \lambda \cdot x$ for a number $\lambda \in \mathbb{R}$

$\Longleftrightarrow (A - \lambda \mathbb{1})x = 0$ for a number $\lambda \in \mathbb{R}$

$\Longleftrightarrow x \in \operatorname{Ker}(A - \lambda \mathbb{1})$ for a number $\lambda \in \mathbb{R}$

eigenvector (if $x \neq 0$) eigenvalue

Example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longleftrightarrow \begin{array}{rl} x_1 + x_2 = \lambda \cdot x_1 & \mathrm{I} \\ x_2 = \lambda \cdot x_2 & \mathrm{II} \end{array}$$

For II: $\boxed{\lambda = 1}$ or $\underbrace{x_2 = 0}_{\mathrm{I}}$ $\Rightarrow x_1 = \lambda \cdot x_1 \Rightarrow \boxed{\lambda = 1}$ or $x_1 = 0$

For I: $x_1 + x_2 = x_1 \Rightarrow x_2 = 0$

Solution: eigenvalue: $\lambda = 1$

eigenvectors: $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ for $x_1 \in \mathbb{R} \setminus \{0\}$

Definition: $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$.

If there is $x \in \mathbb{R}^n \setminus \{0\}$ with $Ax = \lambda x$, then:

- λ is called an eigenvalue of A
- x is called an eigenvector of A (associated to λ)
- $\operatorname{Ker}(A - \lambda \mathbb{1})$ eigenspace of A (associated to λ)

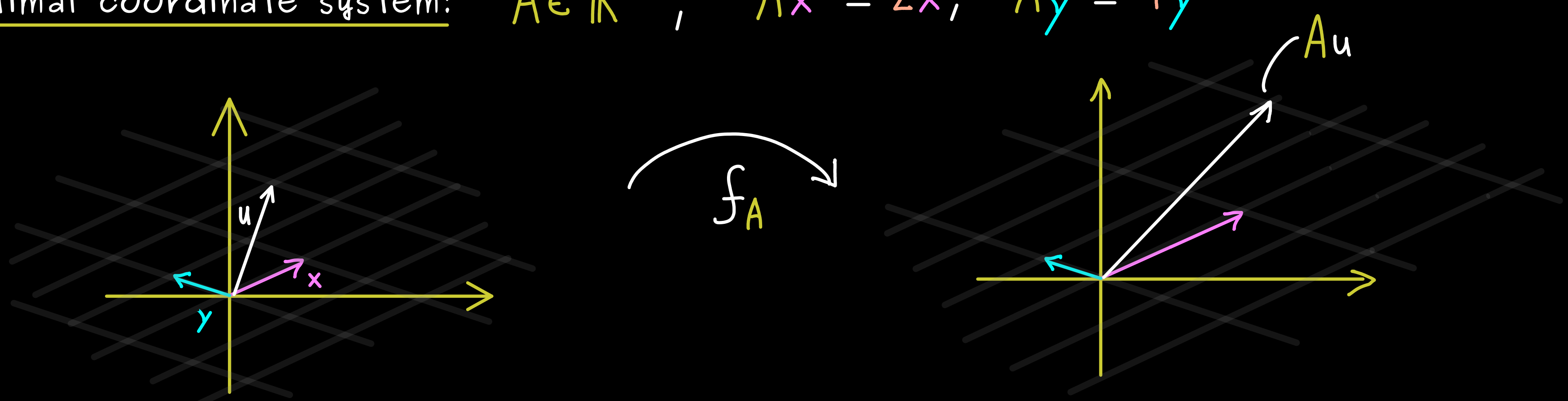
The set of all eigenvalues of A : $\operatorname{spec}(A)$ spectrum of A

Linear Algebra – Part 54

$A \in \mathbb{R}^{n \times n} \quad \leftrightsquigarrow \quad f_A : \mathbb{R}^n \to \mathbb{R}^n$ linear map

eigenvalue equation: $Ax = \lambda \cdot x \;, \quad x \neq 0$

optimal coordinate system: $A \in \mathbb{R}^{2\times 2}, \quad Ax = 2x, \quad Ay = 1y$



$u = a \cdot x + b \cdot y$

$$\begin{aligned} Au &= A(a\cdot x + b \cdot y) \\ &= a \cdot Ax + b\, Ay \\ &= 2ax + 1by \end{aligned}$$

How to find enough eigenvectors?

$x \neq 0$ eigenvector associated to eigenvalue $\lambda \iff x \in \mathrm{Ker}(\underbrace{A - \lambda \mathbb{1}}_{\text{singular matrix}})$

$\det(A - \lambda \mathbb{1}) = 0 \iff \mathrm{Ker}(A - \lambda\mathbb{1})$ is non-trivial

$\iff \lambda$ is eigenvalue of A

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad A - \lambda \mathbb{1} = \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix}$

$$\begin{aligned} \det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} &= (3-\lambda)(4-\lambda) - 2 \\ &= 10 - 7\lambda + \lambda^2 \\ &= (\lambda - 5)(\lambda - 2) \overset{!}{=} 0 \end{aligned}$$

characteristic polynomial

$\Longrightarrow$ 2 and 5 are eigenvalues of A

General case: For $A \in \mathbb{R}^{n \times n}$:

$$\det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} - \lambda \end{pmatrix}$$

$$\overset{\text{Leibniz formula}}{=} (a_{11} - \lambda) \cdots (a_{nn} - \lambda) + \cdots$$

$$= (-1)^n \cdot \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda^1 + c_0$$

Definition: For $A \in \mathbb{R}^{n \times n}$, the polynomial of degree n given by

$$p_A : \lambda \longmapsto \det(A - \lambda \mathbb{1})$$

is called the characteristic polynomial of A.

Remember: The zeros of the characteristic polynomial are exactly the eigenvalues of A.

Linear Algebra - Part 55

$$\lambda \in \operatorname{spec}(A) \iff \det(A - \lambda \mathbb{1}) = 0$$

Fundamental theorem of algebra: For $a_n \neq 0$ and $a_n, a_{n-1}, \dots, a_0 \in \mathbb{C}$, we have:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

has n solutions $x_1, x_2, \dots, x_n \in \mathbb{C}$ (not necessarily distinct).

Hence: $p(x) = a_n (x - x_n) \cdot (x - x_{n-1}) \cdots (x - x_1)$

Conclusion for characteristic polynomial: $A \in \mathbb{R}^{n \times n}$, $p_A(\lambda) := \det(A - \lambda \mathbb{1})$

- $p_A(\lambda) = 0$ has at least one solution in $\mathbb{C}$

  $\Longrightarrow$ A has at least one eigenvalue in $\mathbb{C}$

  Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow p_A(\lambda) = \lambda^2 + 1$

  $\Rightarrow$ $-i$ and i are eigenvalues

- $p_A(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

  Example: $A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 2 \end{pmatrix} \Rightarrow p_A(\lambda) = (\lambda - 1)^2 (\lambda - 2)^2$

Definition: If $\tilde{\lambda}$ occurs k times in the factorisation $p_A(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$,

then we say: $\tilde{\lambda}$ has algebraic multiplicity $k =: \alpha(\tilde{\lambda})$

Remember:
- If $\tilde{\lambda} \in \operatorname{spec}(A) \iff 1 \leq \alpha(\tilde{\lambda}) \leq n$
- $\sum_{\tilde{\lambda} \in \mathbb{C}} \alpha(\tilde{\lambda}) = n$

Linear Algebra - Part 56

eigenvalues: $\lambda \in \operatorname{spec}(A) \iff \underbrace{\det(A - \lambda \mathbb{1})}_{\text{characteristic polynomial}} = 0$

Next step for a given $\lambda \in \operatorname{spec}(A)$:

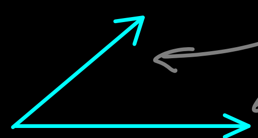
$$\operatorname{Ker}(A - \lambda \mathbb{1}) \supsetneq \{0\}$$

Solve: $$\left(\begin{array}{cccc|c} a_{11} - \lambda & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} - \lambda & & \vdots & 0 \\ \vdots & & \ddots & & \vdots \\ a_{n1} & \cdots & & a_{nn} - \lambda & 0 \end{array}\right)$$

Solution set: eigenspace (associated to λ)

Definition: $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$ eigenvalue

$$\gamma(\lambda) := \dim(\operatorname{Ker}(A - \lambda \mathbb{1}))$$ geometric multiplicity of λ



eigenvectors span eigenspace

Example: $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ characteristic polynomial:

$$\det(A - \lambda \mathbb{1}) = (2-\lambda)(2-\lambda)(3-\lambda) = (2-\lambda)^2 (3-\lambda)$$

$$\implies \operatorname{spec}(A) = \{2, 3\}$$

algebraic multiplicity 2 algebraic multiplicity 1

$$\operatorname{Ker}(A - 2\cdot\mathbb{1}) = \operatorname{Ker}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

solve system: $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{II and III}]{\text{exchange}} \left(\begin{array}{ccc|c} 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{array}{l} x_2 = 0 \\ x_3 = 0 \end{array}$

x_1 free variable

backwards substitution

solution set: $\left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$

eigenvector

$\Longrightarrow$ geometric multiplicity $\gamma(2) = 1 < \alpha(2)$

Linear Algebra - Part 57

Recall:

$\det(A - \lambda \mathbb{1}) = 0$

$\Leftrightarrow$

$\lambda \in \operatorname{spec}(A)$

Proposition:

(a)

$$\operatorname{spec}\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & & & a_{2n} \\ & & \ddots & & \vdots \\ & & & & a_{nn} \end{pmatrix} = \{a_{11}, a_{22}, \dots, a_{nn}\}$$

(b)

$$\operatorname{spec}\begin{pmatrix} \underbrace{\boxed{B}}_{m \times m \text{ matrix}} & C \\ 0 & \underbrace{\boxed{D}}_{k \times k \text{ matrix}} \end{pmatrix} = \operatorname{spec}(B) \cup \operatorname{spec}(D)$$

(part 49)

(c)

$$\operatorname{spec}(A^T) = \operatorname{spec}(A)$$

Example:

(a)

$$\operatorname{spec}\begin{pmatrix} 2 & 5 & 8 & 9 \\ 0 & 3 & 0 & 8 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \{1, 2, 3\}$$

algebraic multiplicity is 2

(b)

$$\operatorname{spec}\begin{pmatrix} 1 & 2 & 4 & 5 & 8 & 7 \\ 0 & 7 & 7 & 9 & 8 & 4 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 \\ 0 & 0 & 5 & 6 & 1 & 2 \\ 0 & 0 & 7 & 9 & 0 & 3 \end{pmatrix} = \operatorname{spec}\begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 5 & 0 & 0 & 0 \\ 7 & 8 & 0 & 0 \\ 5 & 6 & 1 & 2 \\ 7 & 9 & 0 & 3 \end{pmatrix}$$

$$= \{1, 7\} \cup \operatorname{spec}\begin{pmatrix} 5 & 0 \\ 7 & 8 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$= \{1, 7, 5, 8, 1, 3\}$$

$$= \{1, 3, 5, 7, 8\}$$

algebraic multiplicity is 2

Linear Algebra - Part 58

$\operatorname{spec}(A) \subseteq \mathbb{C}$ (fundamental theorem of algebra)

$\hookrightarrow$ Consider $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$

Definition: $\mathbb{C}^n$: column vectors with n entries from $\mathbb{C}$ $\left(\begin{pmatrix} i+2 \\ 1 \end{pmatrix} \in \mathbb{C}^2 \right)$

$\mathbb{C}^{m \times n}$: matrices with $m \times n$ entries from $\mathbb{C}$ $\left(\begin{pmatrix} i & i-1 \\ 0 & 2 \end{pmatrix} \in \mathbb{C}^{2 \times 2} \right)$

Operations like before: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ + in $\mathbb{C}$

$\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$ $\cdot$ in $\mathbb{C}$

Properties: The set $\mathbb{C}^n$ together with $+$, $\cdot$ is a complex vector space:

(a) $(\mathbb{C}^n, +)$ is an abelian group:

(1) $u + (v + w) = (u + v) + w$ (associativity of +)

(2) $v + 0 = v$ with $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ (neutral element)

(3) $v + (-v) = 0$ with $-v = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}$ (inverse elements)

(4) $v + w = w + v$ (commutativity of +)

(b) scalar multiplication is compatible: $\cdot : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$

(5) $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$

(6) $1 \cdot v = v$

(c) distributive laws:

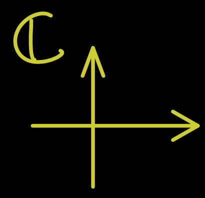
(7) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$

(8) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

$\rightsquigarrow$ same notions: subspace, span, linear independence, basis, dimension,…

Remember: $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \dots, \; e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ basis of $\mathbb{C}^n$

$$\Rightarrow \dim(\mathbb{C}^n) = n \qquad \left(\dim(\mathbb{C}^1) = 1 \right)$$



complex dimension

Standard inner product: $u, v \in \mathbb{C}^n$: $\langle u, v \rangle = \overline{u_1} \cdot v_1 + \overline{u_2} \cdot v_2 + \dots + \overline{u_n} \cdot v_n$

standard norm $\longrightarrow$ $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{|u_1|^2 + \dots + |u_n|^2}$

Example: $\left\| \begin{pmatrix} i \\ -1 \end{pmatrix} \right\| = \sqrt{|i|^2 + |-1|^2} = \sqrt{2}$

Linear Algebra – Part 59

Recall: in $\mathbb{R}^n$: $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$

in $\mathbb{C}^n$: $\langle x, y \rangle = \sum_{k=1}^{n} \overline{x_k} y_k$

in $\mathbb{R}^n$: $\langle x, Ay \rangle = \langle A^T x, y \rangle$

$$\sum_{k=1}^{n} x_k (Ay)_k = \sum_{\substack{k=1 \\ j=1}}^{n} x_k a_{kj} y_j = \sum_{\substack{k=1 \\ j=1}}^{n} (A^T)_{jk} x_k y_j$$

in $\mathbb{C}^n$: $\langle x, Ay \rangle = \sum_{\substack{k=1 \\ j=1}}^{n} \overline{x_k} a_{kj} y_j = \sum_{\substack{k=1 \\ j=1}}^{n} a_{kj} \overline{x_k} y_j = \sum_{\substack{k=1 \\ j=1}}^{n} \overline{\left(\overline{(A^T)_{jk}} x_k\right)} y_j$

$$= \langle A^* x, y \rangle$$

Definition: For $A \in \mathbb{C}^{m \times n}$ with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & & & & \vdots \\ \vdots & & \ddots & & \\ & & & & a_{mn} \end{pmatrix}$,

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \ddots & & \vdots \\ \vdots & & & \\ \overline{a_{1n}} & \cdots & & \overline{a_{mn}} \end{pmatrix} \in \mathbb{C}^{n \times m}$$

is called the adjoint matrix/ conjugate transpose/ Hermitian conjugate.

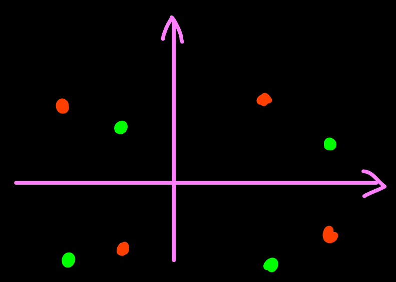
Examples: (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} i & 1+i & 0 \\ 2 & e^{-i} & 1-i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} -i & 2 \\ 1-i & e^{i} \\ 0 & 1+i \end{pmatrix}$

Remember: in $\mathbb{R}^n$: $\langle x, y \rangle = x^T y$ (standard inner product)

in $\mathbb{C}^n$: $\langle x, y \rangle = x^* y$ (standard inner product)

Proposition: $\operatorname{spec}(A^*) = \{ \overline{\lambda} \mid \lambda \in \operatorname{spec}(A) \}$



Linear Algebra – Part 60

Definition: A complex matrix $A \in \mathbb{C}^{n \times n}$ is called:

(1) selfadjoint if $A^* = A$

(2) skew-adjoint $A^* = -A$

(3) unitary if $A^*A = AA^* = \mathbb{1}$ (=identity matrix)

(4) normal if $A^*A = AA^*$

Example: (a) $A = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{1} & \overline{-2i} \\ \overline{2i} & \overline{0} \end{pmatrix} = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} = A$

(b) $A = \begin{pmatrix} i & -1+2i \\ 1+2i & 3i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{i} & \overline{1+2i} \\ \overline{-1+2i} & \overline{3i} \end{pmatrix} = \begin{pmatrix} -i & 1-2i \\ -1-2i & -3i \end{pmatrix} = -A$

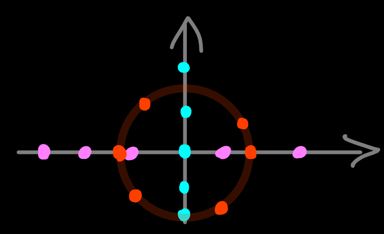
(c) $A = \begin{pmatrix} i & 0 \\ 0 & 4 \end{pmatrix}$ not selfadoint nor skew-adjoint but normal.

Remember:

$A \in \mathbb{C}^{n \times n}$	$A \in \mathbb{R}^{n \times n}$
adjoint A^*	transpose A^T
selfadjoint	symmetric
skew-adjoint	skew-symmetric
unitary	orthogonal

Proposition:

(a) A selfadjoint $\Longrightarrow$ $\operatorname{spec}(A) \subseteq$ real axis



(b) A skew-adjoint $\Longrightarrow$ $\operatorname{spec}(A) \subseteq$ imaginary axis

(c) A unitary $\Longrightarrow$ $\operatorname{spec}(A) \subseteq$ unit circle

Proof: (a) $\lambda \in \operatorname{spec}(A) \Longrightarrow$ eigenvalue equation $Ax = \lambda x$, $x \neq 0$, $\boxed{\text{choose: } \|x\| = 1}$

$$\lambda \cdot \underbrace{\langle x, x \rangle}_{1} = \langle x, \lambda \cdot x \rangle = \langle x, A x \rangle = \langle A^{*} x, x \rangle \overset{\text{selfadjoint}}{=} \langle A x, x \rangle = \langle \lambda \cdot x, x \rangle = \overline{\lambda} \underbrace{\langle x, x \rangle}_{= 1}$$

(c)

$\lambda \in \operatorname{spec}(A) \Longrightarrow$ eigenvalue equation $Ax = \lambda x$, $x \neq 0$, $\boxed{\text{choose: } \|x\| = 1}$

$$\langle \lambda x, \lambda x \rangle = \langle A x, A x \rangle = \langle \underbrace{A^{*} A}_{\mathbb{1}} x, x \rangle = \langle x, x \rangle = 1$$

$$\|$$

$$\overline{\lambda} \cdot \lambda \langle x, x \rangle = |\lambda|^{2} \Longrightarrow \lambda \text{ lies on the unit circle} \qquad \square$$

Linear Algebra - Part 61

Definition: $A, B \in \mathbb{C}^{n\times n}$ are called similar if there is an invertible $S \in \mathbb{C}^{n\times n}$ such that $A = S^{-1} B S$.

(For similiar matrices: f_A injective $\Longleftrightarrow$ f_B injective)

(For similiar matrices: f_A surjective $\Longleftrightarrow$ f_B surjective)

change of basis

Property: Similar matrices have the same characteristic polynomial.

Hence: A, B similar $\Longrightarrow$ $\mathrm{spec}(A) = \mathrm{spec}(B)$

Proof:

$$p_A(\lambda) = \det(A - \lambda \mathbb{1}) = \det(S^{-1} B S - \lambda \mathbb{1}) = \det(S^{-1}(B - \lambda\mathbb{1}) S)$$

$$= \underbrace{\det(S^{-1}) \det(B - \lambda \mathbb{1}) \det(S)}_{\det(S^{-1})\det(S) = \det(\mathbb{1}) = 1} = p_B(\lambda)$$

Later:

- A normal $\Longrightarrow$ $A = S^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} S$ (eigenvalues on the diagonal)

- $A \in \mathbb{C}^{n\times n}$ $\Longrightarrow$ $A = S^{-1} \begin{pmatrix} \lambda_1 & & (*) \\ & \ddots & \\ & & \lambda_n \end{pmatrix} S$ (eigenvalues on the diagonal)

(Jordan normal form)

Linear Algebra - Part 62

Recall: $\alpha(\lambda)$ algebraic multiplicity

$\gamma(\lambda)$ geometric multiplicity (= dimension of $\mathrm{Eig}(\lambda)$)

Recipe: $A \in \mathbb{C}^{n\times n}$: (1) Calculate the zeros of $p_A(\lambda) = \det(A - \lambda \mathbb{1})$.

Call them $\lambda_1, \dots, \lambda_k$,

with $\underbrace{\alpha(\lambda_1), \dots, \alpha(\lambda_k)}_{\text{sum is equal to } n}$.

$\left[A \in \mathbb{R}^{n\times n}, \quad \lambda_j \text{ zero of } p_A \Rightarrow \overline{\lambda_j} \text{ zero of } p_A \right]$

(2) For $j \in \{1, \dots, k\}$: Solve LES $(A - \lambda_j \mathbb{1})x = 0$

Solution set: $\mathrm{Eig}(\lambda_j)$ (eigenspace)

(3) All eigenvectors: $\bigcup_{j=1}^{k} \mathrm{Eig}(\lambda_j) \setminus \{0\}$

Example: $A = \begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}$

(1) $p_A(\lambda) = \det \begin{pmatrix} 8-\lambda & 8 & 4 \\ -1 & 2-\lambda & 1 \\ -2 & -4 & -2-\lambda \end{pmatrix}$

Sarrus

$$= (8-\lambda)(2-\lambda)(-2-\lambda) + 16 - 16 + 8(2-\lambda) + 4(8-\lambda) + 8(-2-\lambda)$$

$$= (8-\lambda)(-4+\lambda^2) + 16 - 8\lambda + 32 - 4\lambda - 16 - 8\lambda$$

$$= (8-\lambda)(-4+\lambda^2) - 20\lambda + 32$$

$$= -32 + 4\lambda + 8\lambda^2 - \lambda^3 - 20\lambda + 32$$

$$= \lambda(-\lambda^2 + 8\lambda - 16) = -\lambda(\lambda-4)^2$$

$p_A(\lambda) = -\lambda^1(\lambda-4)^2$

eigenvalues:

$\lambda_1 = 0$, $\alpha(\lambda_1) = 1$

$\lambda_2 = 4$, $\alpha(\lambda_2) = 2$

(2) eigenspace for $\lambda_1 = 0$

$$\mathrm{Eig}(\lambda_1) = \mathrm{Ker}\,(A-\lambda_1\mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \overset{\mathrm{I}\leftrightarrow\mathrm{II}}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 8 & 8 & 4 \\ -2 & -4 & -2 \end{pmatrix}$$

$$\overset{\substack{\mathrm{II}+8\mathrm{I} \\ \mathrm{III}-2\mathrm{I}}}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 24 & 12 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2}t \\ t \end{pmatrix} \middle| \; t \in \mathbb{C} \right\} = \mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

eigenspace for $\lambda_2 = 4$

$$\mathrm{Eig}(\lambda_2) = \mathrm{Ker}\,(A-\lambda_2\mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 4 & 8 & 4 \\ -1 & -2 & 1 \\ -2 & -4 & -6 \end{pmatrix} \overset{\mathrm{I}\leftrightarrow\mathrm{II}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 4 & 8 & 4 \\ -2 & -4 & -6 \end{pmatrix}$$

$$\overset{\substack{\mathrm{II}+4\mathrm{I} \\ \mathrm{III}-2\mathrm{I}}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix} \overset{\substack{\text{exchange} \\ \text{scale}}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

(3) eigenvectors of A : $\left(\mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \cup \mathrm{Span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right) \setminus \{0\}$

Linear Algebra - Part 63

Assume: x eigenvector for $A \in \mathbb{C}^{n \times n}$ associated to eigenvalue $\lambda \in \mathbb{C}$

Then: $Ax = \lambda x \implies \underset{\overset{\|}{A^2 x}}{A(Ax)} = A(\lambda x) = \lambda(\underbrace{Ax}_{=\lambda x})$

$\implies A^2 x = \lambda^2 x \implies A^3 x = \lambda^3 x$

$\overset{\text{induction}}{\implies} A^m x = \lambda^m x$ for all $m \in \mathbb{N}$

Spectral mapping theorem: $A \in \mathbb{C}^{n \times n}$, $p: \mathbb{C} \to \mathbb{C}$, $p(z) = c_m z^m + \cdots + c_1 z^1 + c_0$

Define: $p(A) = c_m A^m + c_{m-1} A^{m-1} + \cdots + c_1 A + c_0 \cdot \mathbb{1}_n \in \mathbb{C}^{n \times n}$

Then: $\operatorname{spec}(p(A)) = \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$

Proof: Show two inclusion: $(\supseteq)$ (see above) ✓

$(\subseteq)$ $\boxed{\text{1st case:}}$ p constant, $p(z) = c_0$.

Take $\tilde{\lambda} \in \operatorname{spec}(p(A)) \implies \underset{\overset{\|}{(c_0 - \tilde{\lambda})^n}}{\det(\underbrace{p(A)}_{c_0 \mathbb{1}} - \tilde{\lambda}\mathbb{1})} = 0$

$\implies \tilde{\lambda} \in \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$ ✓

$\boxed{\text{2nd case:}}$ p not constant. Do proof by contraposition.

Assume: $\mu \notin \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$

Define polynomial: $q(z) = p(z) - \mu$

$$= \underset{\neq 0}{c} \cdot (z - a_1)(z - a_2) \cdots (z - a_m)$$

By definition of μ : $a_j \notin \operatorname{spec}(A)$ for all j

$$\Longrightarrow \det(A - a_j \mathbb{1}) \neq 0 \quad \text{for all } j$$

Hence: $\det(p(A) - \mu \mathbb{1}) = \det(q(A))$

$$= \det\Big(c \cdot (A - a_1)(A - a_2) \cdots (A - a_m)\Big)$$

$$= c^n \cdot \det\big(A - a_1\big) \det\big(A - a_2\big) \cdots \det\big(A - a_m\big)$$

$$\neq 0$$

$$\Longrightarrow \mu \notin \operatorname{spec}(p(A)) \qquad \square$$

<u>Example:</u> $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, $\operatorname{spec}(A) = \{1, 4\}$

$B = 3A^3 - 7A^2 + A - 2\mathbb{1}$, $\operatorname{spec}(B) = \{-5, 82\}$

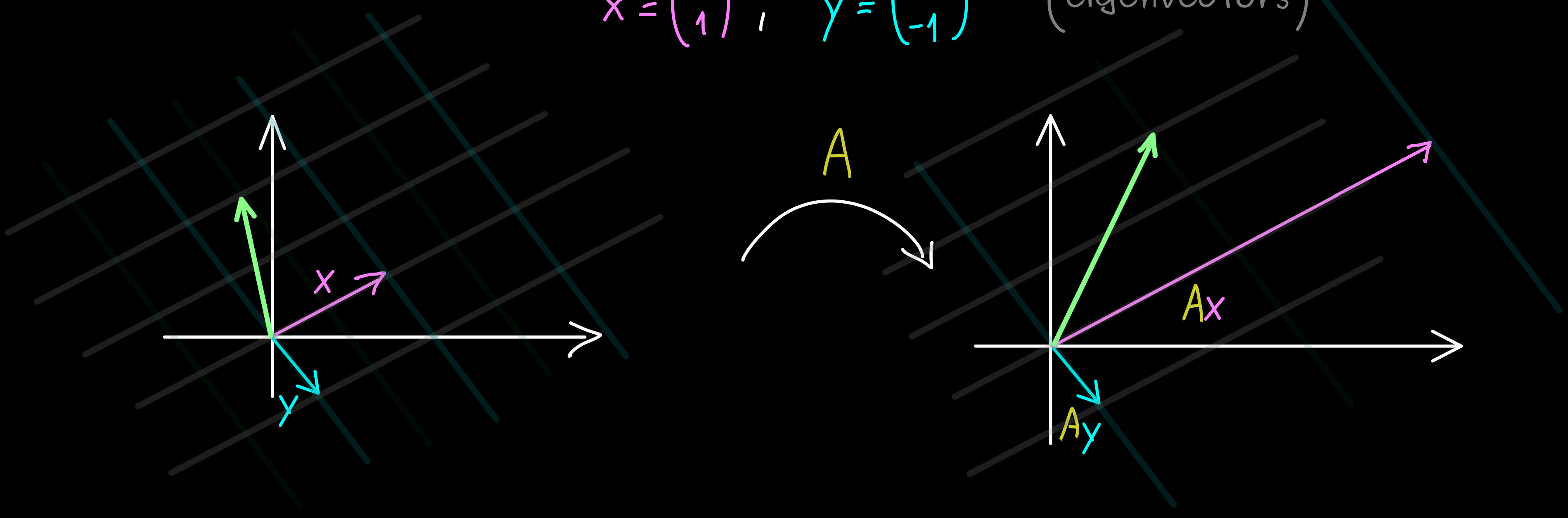
Linear Algebra - Part 64

Diagonalization = transform matrix into a diagonal one

= find a an optimal coordinate system

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, $\lambda_1 = 4$, $\lambda_2 = 1$ (eigenvalues)

$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (eigenvectors)



$$\alpha x + \beta y \longmapsto \alpha \lambda_1 x + \beta \lambda_2 y$$

Diagonalization: $A \in \mathbb{C}^{n \times n} \rightsquigarrow \lambda_1, \lambda_2, \dots, \lambda_n$ (counted with algebraic multiplicities)

$\rightsquigarrow x^{(1)}, x^{(2)}, \dots, x^{(n)}$ (associated eigenvectors)

$\rightsquigarrow A x^{(1)} = \lambda_1 x^{(1)}, \dots, A x^{(n)} = \lambda_n x^{(n)}$ (eigenvalue equations)

$$A \begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Ax^{(1)} & Ax^{(2)} & \cdots & Ax^{(n)} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \lambda_1 x^{(1)} & \lambda_2 x^{(2)} & \cdots & \lambda_n x^{(n)} \\ | & | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} \\ | & | & & | \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}}_{D}$$

$$\Longrightarrow \quad AX = XD$$

If X is invertible, then: $\boxed{D = X^{-1} A X}$ A is similar to a diagonal matrix

Application:

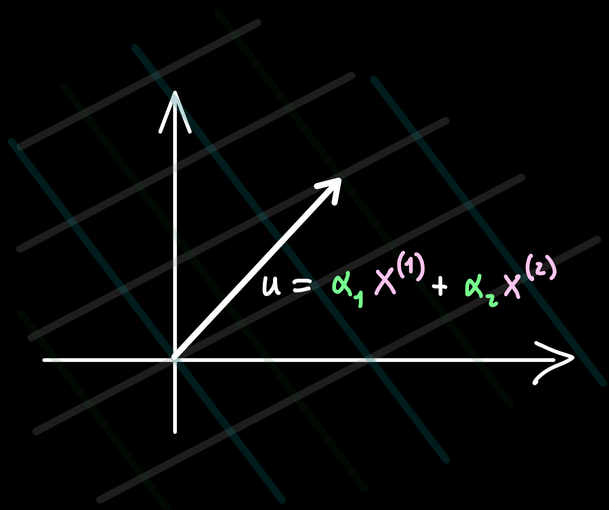
$$A^{98} = \left(X D X^{-1} \right)^{98} = X D \underbrace{X^{-1} X}_{\mathbb{1}} D \underbrace{X^{-1} X}_{\mathbb{1}} D X^{-1} \cdots X D X^{-1} \quad \cdots$$

$$= X D^{98} X^{-1}$$

$$= X \begin{pmatrix} \lambda_1^{98} & & & \\ & \lambda_2^{98} & & \\ & & \ddots & \\ & & & \lambda_n^{98} \end{pmatrix} X^{-1}$$

Linear Algebra - Part 65

$$
\begin{tikzcd}[column sep=huge]
\text{canonical basis:} & u \in \mathbb{C}^n \arrow[r, "A"] \arrow[d, bend left, "X^{-1}"] & Au \in \mathbb{C}^n \arrow[d, bend left, "X^{-1}"] \\
\text{eigenvector basis:} & \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \arrow[r, "D"] \arrow[u, bend left, "X"] & \begin{pmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ \vdots \\ \lambda_n \alpha_n \end{pmatrix} \arrow[u, bend left, "X"]
\end{tikzcd}
$$



$$\boxed{D = X^{-1} A X}$$

Is that possible? For given matrix $A \in \mathbb{C}^{n \times n}$ with eigenvectors $x^{(1)}, x^{(2)}, \dots, x^{(n)}$:

- Can we express each $u \in \mathbb{C}^n$ with $\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \dots + \alpha_n x^{(n)}$?
- $\mathrm{Span}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = \mathbb{C}^n$?
- $(x^{(1)}, x^{(2)}, \dots, x^{(n)})$ basis of $\mathbb{C}^n$?
- $X = \begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \dots & x^{(n)} \\ | & | & & | \end{pmatrix}$ invertible ?

Definition: $A \in \mathbb{C}^{n \times n}$ is called diagonalizable if one can find n eigenvectors of A such that they form a basis $\mathbb{C}^n$.

Example: (a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, e_1, e_2 eigenvectors $\Rightarrow$ A is diagonalizable

(b) $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvectors $\Rightarrow$ B is diagonalizable

(c) $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, all eigenvectors lie in direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ C is <u>not</u> diagonalizable

Remember: For $A \in \mathbb{C}^{n \times n}$:

- $\alpha(\lambda) = \gamma(\lambda)$ for all eigenvalues λ $\Leftrightarrow$ A is diagonalizable
- A normal $\Rightarrow$ A is diagonalizable
  (One can choose even an ONB with eigenvectors)
- A has n different eigenvalues $\Rightarrow$ A is diagonalizable