# Linear Algebra – Part 59

**Recall:** in $\mathbb{R}^n$: $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$

in $\mathbb{C}^n$: $\langle x, y\rangle = \sum_{k=1}^{n} \overline{x_k}\, y_k$

in $\mathbb{R}^n$: $\langle x, Ay\rangle = \langle A^T x, y\rangle$

$$\sum_{k=1}^{n} x_k (Ay)_k = \sum_{\substack{k=1\\ j=1}}^{n} x_k a_{kj} y_j = \sum_{\substack{k=1\\ j=1}}^{n} (A^T)_{jk} x_k y_j$$

in $\mathbb{C}^n$: $\langle x, Ay\rangle = \sum_{\substack{k=1\\ j=1}}^{n} \overline{x_k}\, a_{kj} y_j = \sum_{\substack{k=1\\ j=1}}^{n} a_{kj} \overline{x_k}\, y_j = \sum_{\substack{k=1\\ j=1}}^{n} \overline{\left(\overline{A^T}\right)_{jk} x_k}\; y_j$

$$= \langle A^* x, y\rangle$$

**Definition:** For $A \in \mathbb{C}^{m\times n}$ with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & & \ddots & & \vdots \\ \vdots & & & & a_{mn} \end{pmatrix}$,

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \ddots & & \vdots \\ \vdots & & & \\ \overline{a_{1n}} & \cdots & & \overline{a_{mn}} \end{pmatrix} \in \mathbb{C}^{n\times m}$$

is called the adjoint matrix/ conjugate transpose/ Hermitian conjugate.

**Examples:** (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} i & 1+i & 0 \\ 2 & e^{-i} & 1-i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} -i & 2 \\ 1-i & e^{i} \\ 0 & 1+i \end{pmatrix}$

**Remember:** in $\mathbb{R}^n$: $\langle x, y\rangle = x^T y$ (standard inner product)

in $\mathbb{C}^n$: $\langle x, y\rangle = x^* y$ (standard inner product)

**Proposition:** $\mathrm{spec}(A^*) = \{\, \overline{\lambda} \mid \lambda \in \mathrm{spec}(A) \,\}$