Linear Algebra – Part 58

$\mathrm{spec}(A) \subseteq \mathbb{C}$ (fundamental theorem of algebra)

$\hookrightarrow$ Consider $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$

Definition: $\mathbb{C}^n$: column vectors with n entries from $\mathbb{C}$ $\left(\begin{pmatrix} i+2 \\ 1 \end{pmatrix} \in \mathbb{C}^2 \right)$

$\mathbb{C}^{m \times n}$: matrices with $m \times n$ entries from $\mathbb{C}$ $\left(\begin{pmatrix} i & i-1 \\ 0 & 2 \end{pmatrix} \in \mathbb{C}^{2\times 2} \right)$

Operations like before:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \quad + \text{ in } \mathbb{C}$$

$$\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} \quad \cdot \text{ in } \mathbb{C}$$

Properties: The set $\mathbb{C}^n$ together with $+ , \cdot$ is a complex vector space:

(a) $(\mathbb{C}^n, +)$ is an abelian group:

(1) $u + (v + w) = (u + v) + w$ (associativity of $+$)

(2) $v + 0 = v$ with $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ (neutral element)

(3) $v + (-v) = 0$ with $-v = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}$ (inverse elements)

(4) $v + w = w + v$ (commutativity of $+$)

(b) scalar multiplication is compatible: $\cdot : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$

(5) $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$

(6) $1 \cdot v = v$

(c) distributive laws:

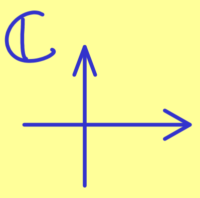
(7) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$

(8) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

$\rightsquigarrow$ same notions: subspace, span, linear independence, basis, dimension,...

Remember: $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \dots, \; e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ basis of $\mathbb{C}^n$

$$\Longrightarrow \dim(\mathbb{C}^n) = n \qquad \left(\dim(\mathbb{C}^1) = 1 \right)$$



complex dimension

Standard inner product: $u, v \in \mathbb{C}^n$: $\langle u, v \rangle = \overline{u_1} \cdot v_1 + \overline{u_2} \cdot v_2 + \dots + \overline{u_n} \cdot v_n$

standard norm $\longrightarrow$ $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{|u_1|^2 + \dots + |u_n|^2}$

Example: $\left\| \begin{pmatrix} i \\ -1 \end{pmatrix} \right\| = \sqrt{|i|^2 + |-1|^2} = \sqrt{2}$