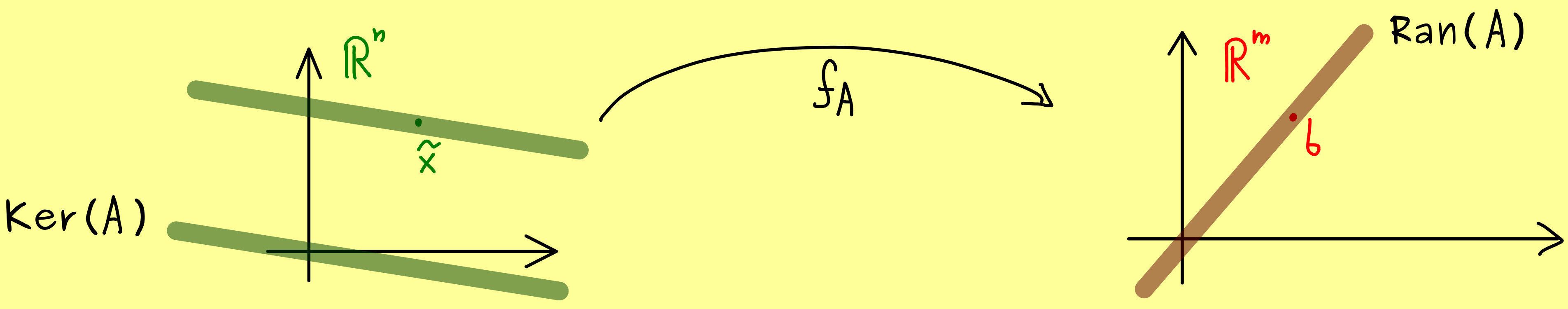
The Bright Side of Mathematics

Linear Algebra – Part 38

Set of solutions: $Ax = b \quad (A \in \mathbb{R}^{m \times n})$

solution: $\tilde{x}$ satisfies $A\tilde{x} = b$



uniqueness needs $\mathrm{Ker}(A) = \{0\}$

existence needs $b \in \mathrm{Ran}(A)$

Proposition: For a system $Ax = b \quad (A \in \mathbb{R}^{m \times n})$

the set of solutions $S := \{ \tilde{x} \in \mathbb{R}^n \mid A\tilde{x} = b \}$

is an affine subspace (or empty).

More concretely: We have either $S = \emptyset$

or $S = v_0 + \mathrm{Ker}(A)$ for a vector $v_0 \in \mathbb{R}^n$

$= \{ v_0 + x_0 \mid x_0 \in \mathrm{Ker}(A) \}$

Proof: Assume $v_0 \in S$. $\Rightarrow A v_0 = b$

Set $\tilde{x} := v_0 + x_0$ for a vector $x_0 \in \mathbb{R}^n$.

Then: $\tilde{x} \in S \iff A\underbrace{\tilde{x}}_{(v_0 + x_0)} = b \iff \underbrace{A v_0}_{b} + A x_0 = b$

$\iff A x_0 = 0 \iff x_0 \in \mathrm{Ker}(A)$ $\square$

Remember: Row operations don't change the set of solutions!

$S = v_0 + \mathrm{Ker}(A)$

$= \mathrm{Ker}(MA)$

$A v_0 = b$

$\iff MA v_0 = Mb$

$\rightsquigarrow$ Gaussian elimination

- decide $b \in \mathrm{Ran}(A)$
- gives us a particular solution v_0
- gives us $\mathrm{Ker}(A)$