BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Linear Algebra – Part 11

Matrices ⇝ help us to solve systems of linear equations

Matrix = table of numbers

$$a_{ij} \in \mathbb{R} \qquad \left.\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\text{width} = n \text{ columns}}\right\} \text{height} = m \text{ rows}$$

Example: $n = 3$, $m = 2$ $\quad \begin{pmatrix} 4 & \pi & 1 \\ 6 & \sqrt{2} & 0 \end{pmatrix}$

Set of matrices: $\mathbb{R}^{m \times n}$

↳ addition and scalar multiplication ⇝ vector space

Addition: $A, B \in \mathbb{R}^{m \times n}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$

Note: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$ is not defined!

Scalar multiplication: $A \in \mathbb{R}^{m \times n}$, $\lambda \in \mathbb{R}$

$$\lambda \cdot A = \lambda \cdot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} \lambda \cdot a_{11} & \cdots & \lambda \cdot a_{1n} \\ \vdots & & \vdots \\ \lambda \cdot a_{m1} & \cdots & \lambda \cdot a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

↳ $(\mathbb{R}^{m \times n}, +, \cdot)$ is a vector space

Properties: (a) $(\mathbb{R}^{m \times n}, +)$ is an abelian group:

(1) $A + (B + C) = (A + B) + C$ (associativity of +)

(2) $A + 0 = A$ with $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ (neutral element)

(3) $A + (-A) = 0$ with $-A = \begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{pmatrix}$ (inverse elements)

(4) $A + B = B + A$ (commutativity of +)

(b) scalar multiplication is compatible: $\cdot : \mathbb{R} \times \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$

(5) $\lambda \cdot (\mu \cdot A) = (\lambda \cdot \mu) \cdot A$

(6) $1 \cdot A = A$

(c) distributive laws:

(7) $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$

(8) $(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$