The Bright Side of Mathematics

The following pages cover the whole Linear Algebra course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Linear Algebra – Part 1

Variables
a, b, c, d
Unknowns:
x, y, z

+

simple relations between the variables

Linear Algebra $\longrightarrow$

Find all solutions for
x, y, z

Abstract level: vector space | linear maps

Concrete level: $\mathbb{R}^n$ | matrices

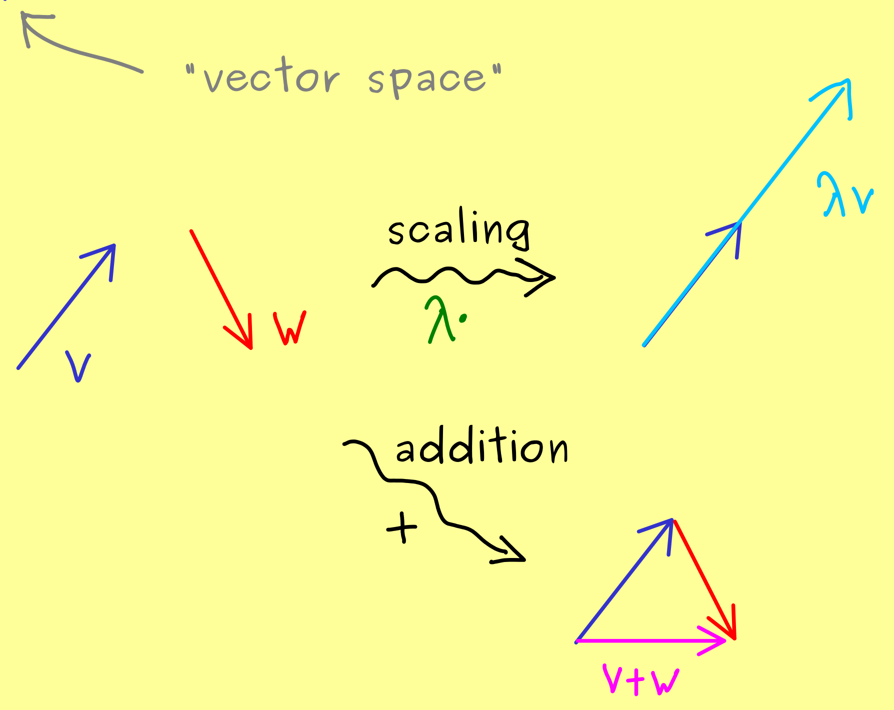
Prerequisites: Start Learning Mathematics (logical symbols, set operations, maps…)

Linear Algebra – Part 2

Vectors in $\mathbb{R}^2$

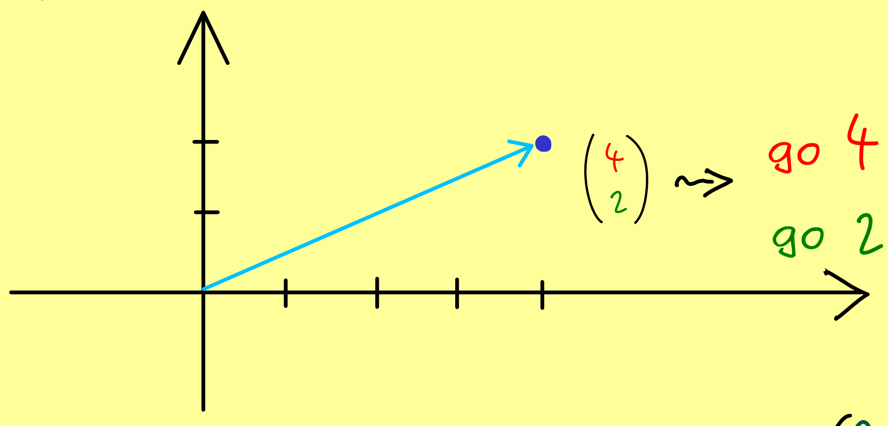
What are vectors? "vector space"

Calculation rules visualised:



Definition: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, elements written in column form: $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \in \mathbb{R}^2$

(Cartesian product)



$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \rightsquigarrow$ go 4 steps to the right

go 2 steps to the top

Scaling: $\lambda \in \mathbb{R}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$: $\lambda \cdot v := \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}$

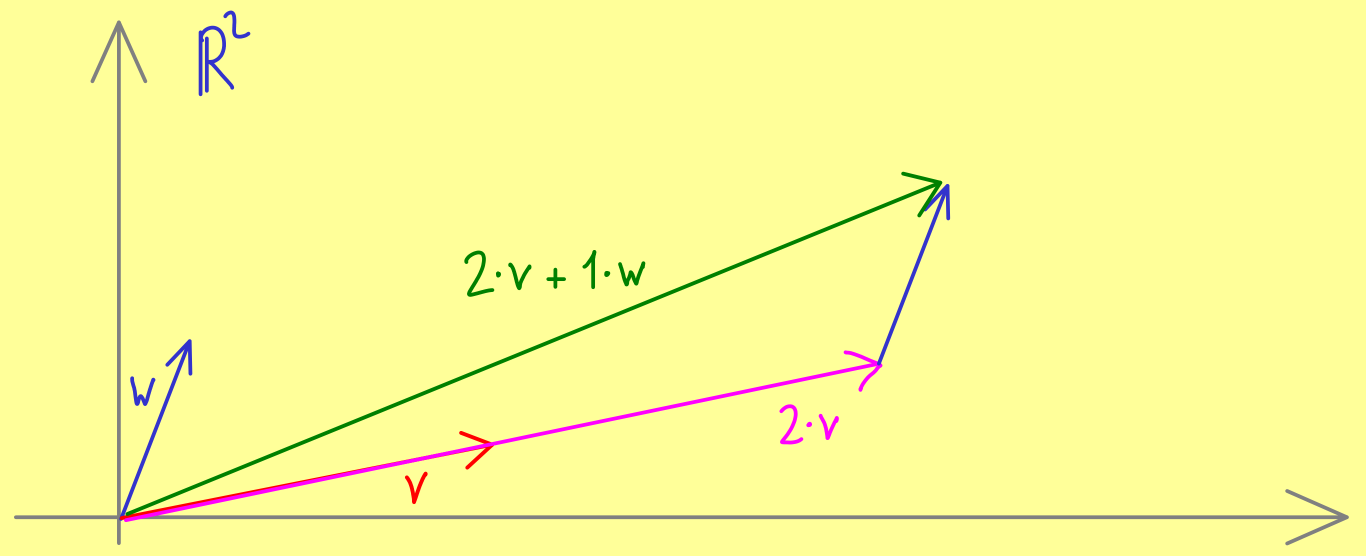
Addition: $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$: $v + w := \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$

$\mathbb{R}^2$ together with the two operations $(\cdot, +)$ is called the vector space $\mathbb{R}^2$

Linear Algebra – Part 3

$\mathbb{R}^2$ with two operations $(\cdot, +)$ is a vector space.

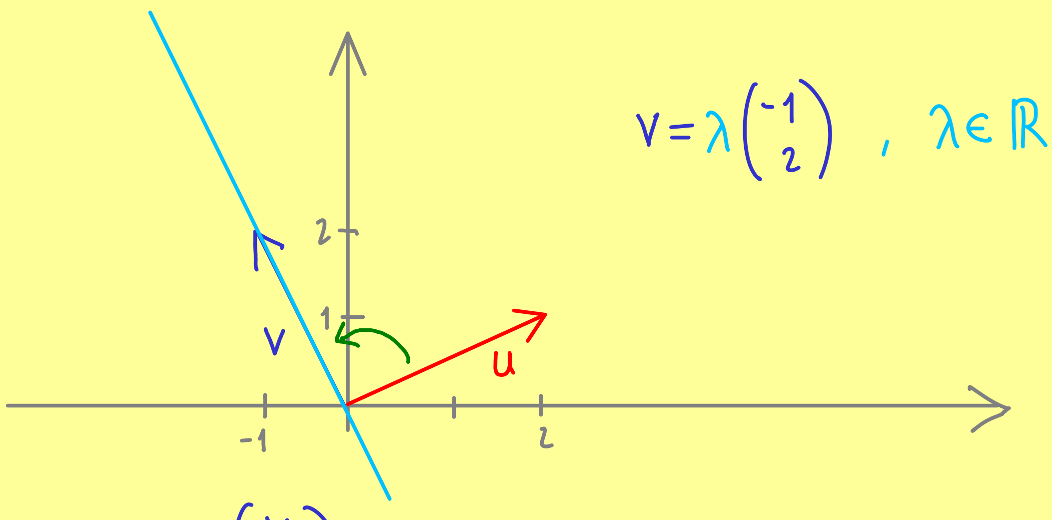
combine them: linear combination



Definition: For vectors $v^{(1)}, v^{(2)}, \dots, v^{(k)} \in \mathbb{R}^2$ and scalars $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$,

the vector $\displaystyle v = \sum_{j=1}^{k} \lambda_j v^{(j)}$ is called a linear combination.

Question: Which vectors $v \in \mathbb{R}^2$ are perpendicular to the vector $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?



$v = \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

Answer: $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ are orthogonal

$\iff \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$\iff u_1 v_1 = -\underbrace{u_1 \lambda}_{v_2} u_2$ and $u_2 v_2 = \underbrace{u_2 \lambda}_{-v_1} \cdot u_1$ for some $\lambda \in \mathbb{R}$

$\iff u_1 v_1 = -v_2 \cdot u_2$ and $u_2 v_2 = -v_1 \cdot u_1$

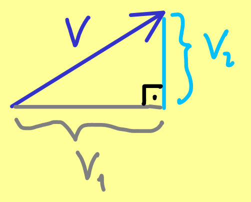
$\iff u_1 v_1 + u_2 v_2 = 0$

$\quad /\!/$

$\langle u, v \rangle$ (standard) inner product

more structure (geometry)

Definition:



length of $v = \sqrt{v_1^2 + v_2^2}$

$\|v\| := \sqrt{\langle v, v \rangle}$ is called the (standard) norm

Euclidean $/\!/$ (standard)

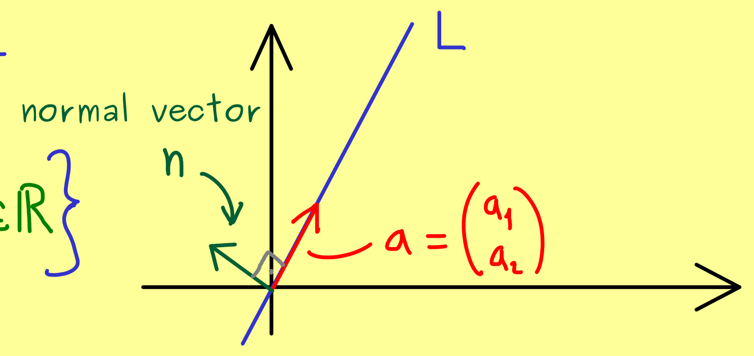
The Bright Side of Mathematics

Linear Algebra – Part 4

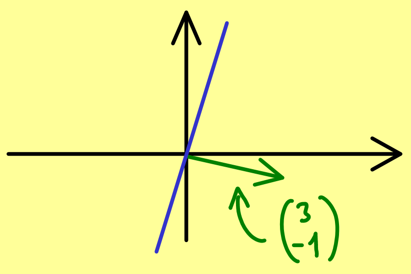
1st case: origin on the line L

$$L = \{v\in\mathbb{R}^2 \mid v = \lambda\cdot a \text{ for } \lambda\in\mathbb{R}\}$$

$$= \{v\in\mathbb{R}^2 \mid \langle n, v\rangle = 0\}$$



Example:



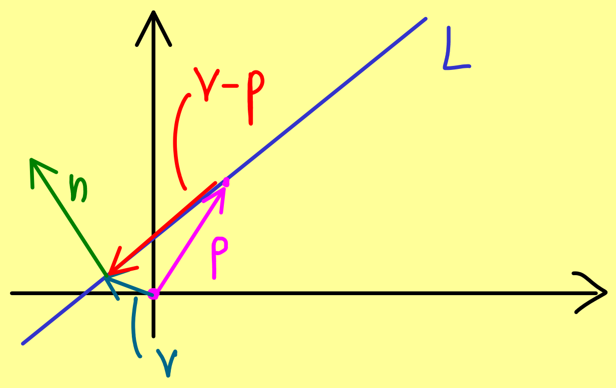
$$L = \left\{\binom{x}{y}\in\mathbb{R}^2 \mid \left\langle \binom{3}{-1}, \binom{x}{y}\right\rangle = 0\right\}$$

$$= \left\{\binom{x}{y}\in\mathbb{R}^2 \mid y = 3x\right\}$$

2nd case: origin not on line L

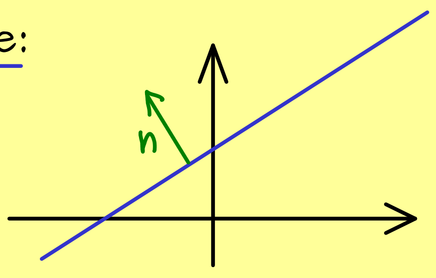
$$L = \{v\in\mathbb{R}^2 \mid \langle n, v-p\rangle = 0\}$$

$$= \left\{\binom{x}{y}\in\mathbb{R}^2 \mid n_1 x + n_2\cdot y = \delta\right\}$$



$$\delta := \langle n, p\rangle$$

Example:



$$L = \left\{\binom{x}{y}\in\mathbb{R}^2 \mid \underbrace{y = 2x+5}_{-2x+y=5}\right\}$$

$$n = \binom{-2}{1}$$

$$\delta = 5$$

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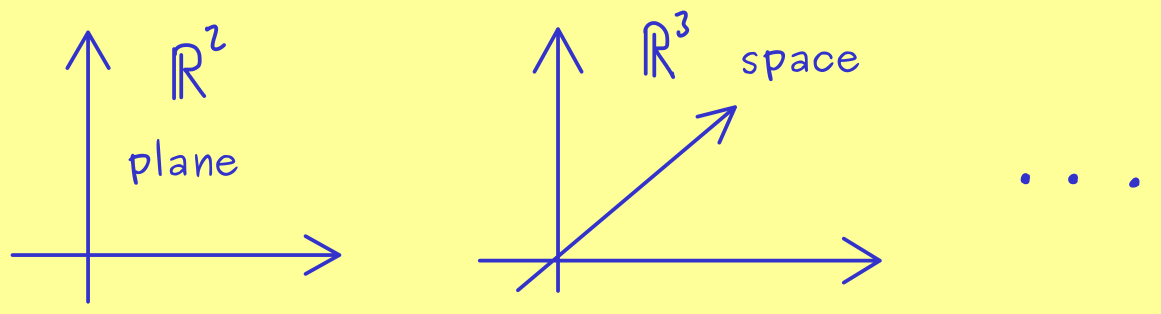
Linear Algebra – Part 5



$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \quad \text{for} \quad n \in \mathbb{N}$$

write $v \in \mathbb{R}^n$ in column form: $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

<u>addition:</u> $u + v = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$

<u>scalar multiplication:</u> $\lambda \cdot u = \lambda \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} := \begin{pmatrix} \lambda \cdot u_1 \\ \vdots \\ \lambda \cdot u_n \end{pmatrix}$

$\hookrightarrow (\mathbb{R}^n, +, \cdot)$ is a <u>vector space</u>

<u>Properties:</u> (a) $(\mathbb{R}^n, +)$ is an abelian group:

(1) $u + (v + w) = (u + v) + w$ (associativity of $+$)

(2) $v + 0 = v$ with $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ (neutral element)

(3) $v + (-v) = 0$ with $-v = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}$ (inverse elements)

(4) $v + w = w + v$ (commutativity of $+$)

(b) scalar multiplication is compatible: $\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

(5) $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$

(6) $1 \cdot v = v$

(c) distributive laws:

(7) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$

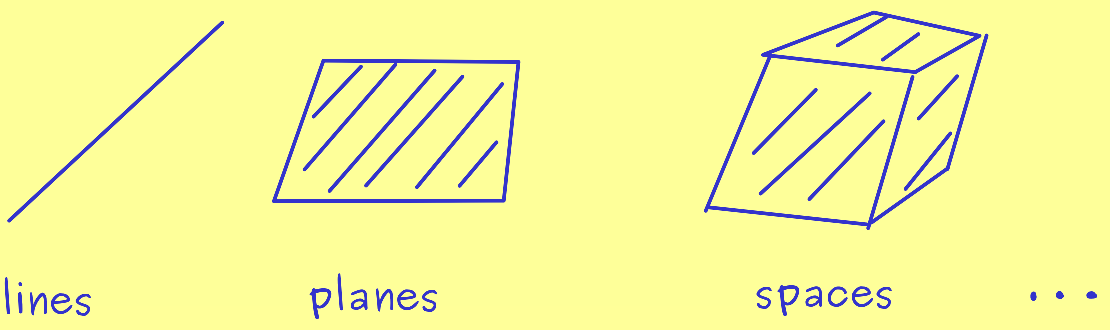
(8) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

<u>Canonical unit vectors:</u> $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ can be written as a linear combination: $v = \sum_{j=1}^{n} v_j \cdot e_j$

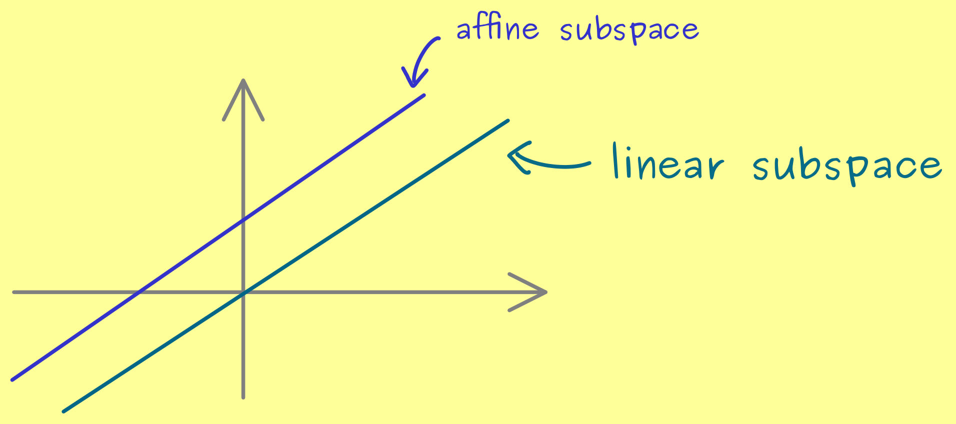
Linear Algebra – Part 6

(linear) subspaces:



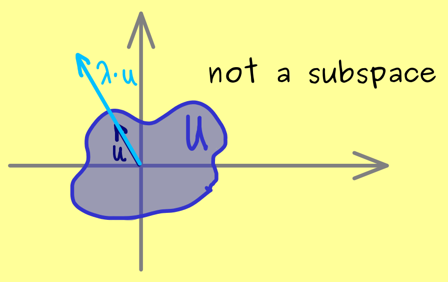
with special properties

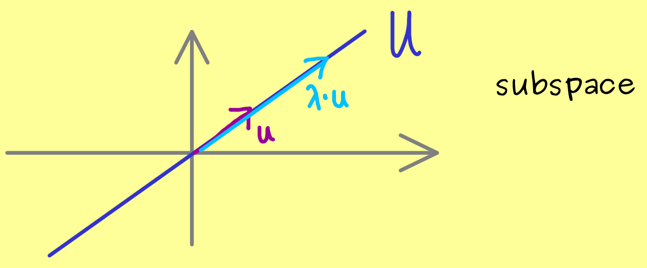
In $\mathbb{R}^2$:



Definition: $U \subseteq \mathbb{R}^n$, $U \neq \emptyset$, is called a (linear) subspace of $\mathbb{R}^n$ if all linear combinations in U remain in U:

$$\left.\begin{array}{l} u^{(1)}, u^{(2)}, \ldots, u^{(k)} \in U \\ \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R} \end{array}\right. \Longrightarrow \sum_{j=1}^{k} \lambda_j u^{(j)} \in U$$





Characterisation for subspaces:

$U \subseteq \mathbb{R}^n$ is a subspace $\Longleftrightarrow$

(a) $0 \in U$

(b) $u \in U, \lambda \in \mathbb{R} \Longrightarrow \lambda \cdot u \in U$

(c) $u, v \in U \Longrightarrow u + v \in U$

Examples: $U = \{0\}$ subspace!

$U = \mathbb{R}^n$

all other subspaces U satisfy: $\{0\} \subsetneq U \subsetneq \mathbb{R}^n$

Linear Algebra – Part 7

Examples for subspaces: (1) $\mathcal{U} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = x_2 \text{ and } x_3 = -2x_2 \right\}$

Is this a subspace?

Checking: (a) Is the zero vector in $\mathcal{U}$?

$$x = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 = x_2 \\ x_3 = 0 = -2x_2 \end{matrix}$$

$$\Rightarrow 0 \in \mathcal{U} \checkmark$$

(b) Is $\mathcal{U}$ closed under scalar multiplication?

Assume: $u \in \mathcal{U}$, $\lambda \in \mathbb{R}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

Then: $u_1 = u_2$, $u_3 = -2u_2$

What about? $x := \lambda \cdot u$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix}$

Do we have? $x_1 = x_2$, $x_3 = -2x_2$ which is equivalent to $\lambda u_1 = \lambda u_2$, $\lambda u_3 = -2 \cdot (\lambda u_2)$

Proof: $\begin{matrix} u_1 = u_2 \\ u_3 = -2u_2 \end{matrix} \overset{\lambda\cdot}{\Rightarrow} \begin{matrix} \lambda u_1 = \lambda u_2 \\ \lambda u_3 = -2(\lambda u_2) \end{matrix} \Rightarrow x := \lambda \cdot u \in \mathcal{U} \checkmark$

(c) Is $\mathcal{U}$ closed under vector addition?

Assume: $u, v \in \mathcal{U}$, $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Then: $u_1 = u_2$, $u_3 = -2u_2$ and $v_1 = v_2$, $v_3 = -2v_2$

What about? $x := u + v$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$

Do we have? $x_1 = x_2$, $x_3 = -2x_2$ which is equivalent to $u_1 + v_1 = u_2 + v_2$, $u_3 + v_3 = -2(u_2 + v_2)$

Proof: $u_1 = u_2$, $u_3 = -2u_2$ and $v_1 = v_2$, $v_3 = -2v_2$

$$\Rightarrow \begin{matrix} u_1 + v_1 = u_2 + v_2 \\ u_3 + v_3 = -2u_2 + (-2v_2) \end{matrix} \Rightarrow \begin{matrix} u_1 + v_1 = u_2 + v_2 \\ u_3 + v_3 = -2(u_2 + v_2) \end{matrix}$$

$$\Rightarrow x := u + v \in \mathcal{U} \checkmark$$

$\Rightarrow \mathcal{U} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = x_2 \text{ and } x_3 = -2x_2 \right\}$ subspace!

(2) $\mathcal{U} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1^2 = x_2 \right\}$

Show that (b) does not hold: $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{U}$, $\lambda = 2$

What about? $x := \lambda \cdot u = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \notin \mathcal{U}$

$4 = 2^2 = x_1^2 \neq x_2 = 2$ $\Rightarrow$ not a subspace!

BECOME A MEMBER

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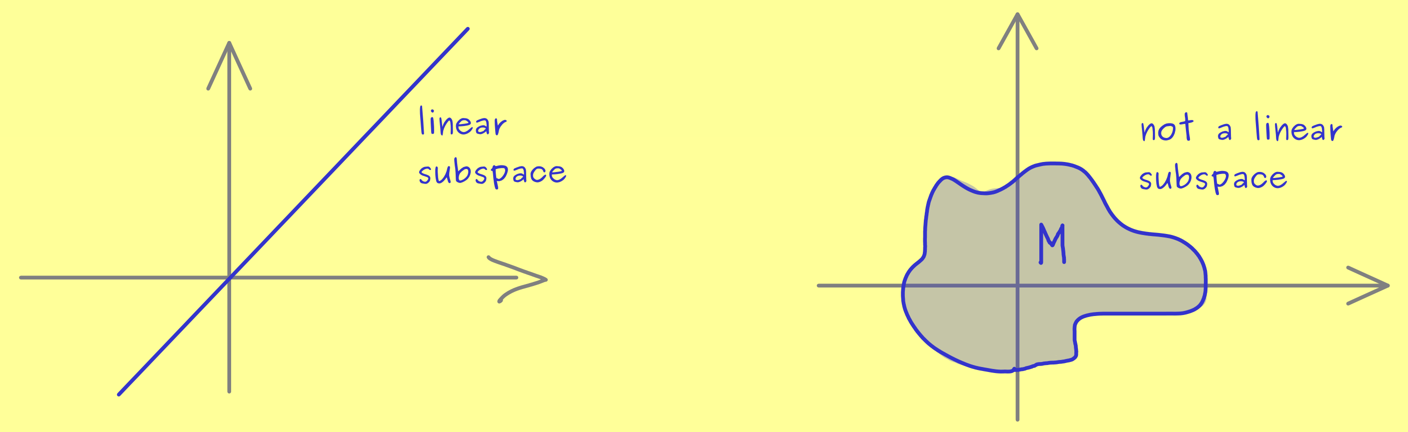
Linear Algebra – Part 8

linear span/ linear hull/ span



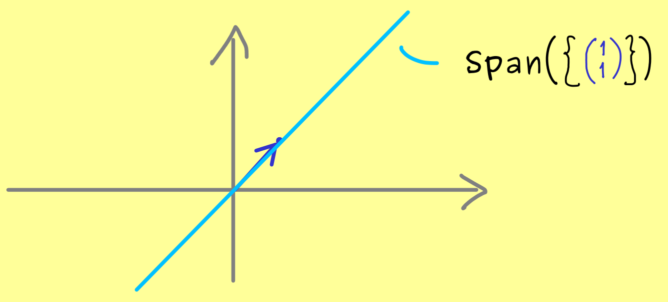
Span(M)
- linear subspace
- contains all linear combinations of vectors from M
- smallest subspace with this property

Definition: $M \subseteq \mathbb{R}^n$ non-empty

$$\text{Span}(M) := \left\{ u \in \mathbb{R}^n \;\middle|\; \text{there are } \lambda_j \in \mathbb{R} \text{ and } u^{(j)} \in M \text{ with: } u = \sum_{j=1}^{k} \lambda_j u^{(j)} \right\}$$

$$\text{Span}(\emptyset) := \{0\}$$

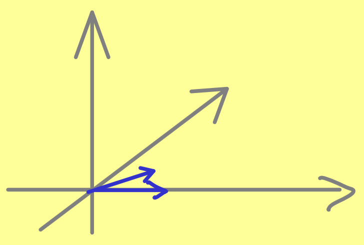
Example: (a) $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$



$$\text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}\right) := \left\{ u \in \mathbb{R}^n \;\middle|\; \text{there is } \lambda \in \mathbb{R} \text{ such that } u = \lambda \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \left\{ \lambda \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{R} \right\} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$



$$\text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

We say: the subspace is generated by the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

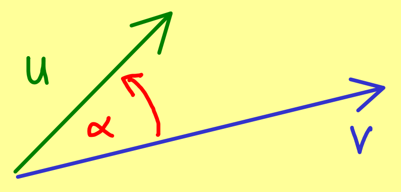
Example: $\mathbb{R}^n = \text{Span}(e_1, e_2, \dots, e_n)$

Linear Algebra – Part 9

inner product and norm in $\mathbb{R}^n$?

↳ give more structure to the vector space

↳ we can do geometry (measure angles and lengths)



Definition: For $u, v \in \mathbb{R}^n$, we define:

$$\langle u, v \rangle := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \quad \text{(standard) inner product}$$

If $\langle u, v \rangle = 0$, we say that u, v are orthogonal.

Properties: The map $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ has the following properties:

(1) $\langle u, u \rangle \geq 0$ for all $u \in \mathbb{R}^n$
$\langle u, u \rangle = 0 \iff u = 0$ (positive definite)

(2) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in \mathbb{R}^n$ (symmetric)

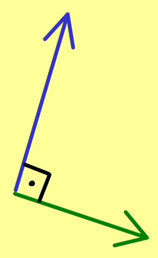
(3) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
$\langle u, \lambda \cdot v \rangle = \lambda \cdot \langle u, v \rangle$ (linear in the 2nd argument)

for all $u, v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Definition: For $u \in \mathbb{R}^n$, we define:

$$\|u\| := \sqrt{\langle u, u \rangle} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \quad \text{(standard) norm} \;\; (= \text{Euclidean})$$

Example: $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4$, $v = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4$, $\langle u, v \rangle = 0$



$\|u\| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\quad \|v\| = \sqrt{2^2} = 2$

Linear Algebra – Part 10

Cross product / vector product

↳ only $\mathbb{R}^3$

map $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

Definition: For $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$, we define the cross product:

$$u \times v = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

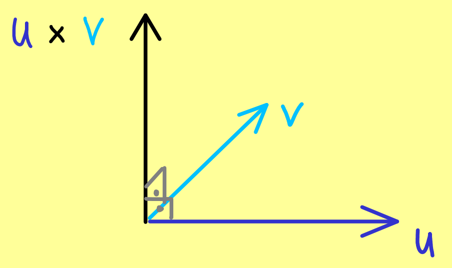
With Levi-Civita symbol: $u \times v = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \, u_i \, v_j \, e_k$

canonical unit vector

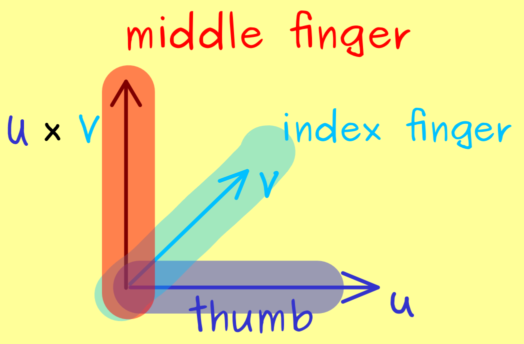
Properties: (1) orthogonality: $u \times v$ orthogonal to u

$u \times v$ orthogonal to v

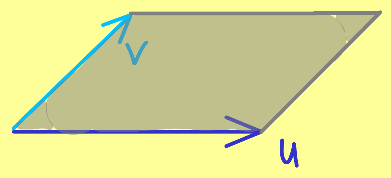
(with respect to the standard inner product)



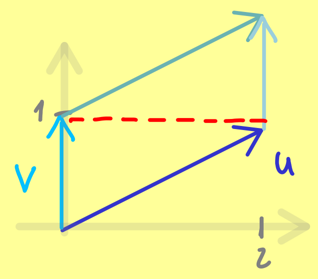
(2) orientation: right-hand rule



(3) length: $\| u \times v \|$ = area of the parallelogram



Example: $u = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$



$$u \times v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 - 0 \cdot 1 \\ 0 \cdot 0 - 2 \cdot 0 \\ 2 \cdot 1 - 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

(1) orthogonality ✓

(2) right-hand rule ✓

(3) length ✓

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Linear Algebra – Part 11

Matrices $\rightsquigarrow$ help us to solve systems of linear equations

Matrix = table of numbers

$$a_{ij} \in \mathbb{R} \qquad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\text{width} = n \text{ columns}} \left.\vphantom{\begin{pmatrix} a \\ a \\ a \\ a \end{pmatrix}}\right\} \text{height} = m \text{ rows}$$

Example: $n = 3$, $m = 2$ $\qquad \begin{pmatrix} 4 & \pi & 1 \\ 6 & \sqrt{2} & 0 \end{pmatrix}$

Set of matrices: $\mathbb{R}^{m \times n}$

$\hookrightarrow$ addition and scalar multiplication $\rightsquigarrow$ vector space

Addition: $A, B \in \mathbb{R}^{m \times n}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$

Note: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$ is not defined!

Scalar multiplication: $A \in \mathbb{R}^{m \times n}$, $\lambda \in \mathbb{R}$

$$\lambda \cdot A = \lambda \cdot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} := \begin{pmatrix} \lambda \cdot a_{11} & \cdots & \lambda \cdot a_{1n} \\ \vdots & & \vdots \\ \lambda \cdot a_{m1} & \cdots & \lambda \cdot a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$\hookrightarrow (\mathbb{R}^{m \times n}, +, \cdot)$ is a vector space

Properties: (a) $(\mathbb{R}^{m \times n}, +)$ is an abelian group:

(1) $A + (B + C) = (A + B) + C$ (associativity of +)

(2) $A + 0 = A$ with $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ (neutral element)

(3) $A + (-A) = 0$ with $-A = \begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{pmatrix}$ (inverse elements)

(4) $A + B = B + A$ (commutativity of +)

(b) scalar multiplication is compatible: $\cdot : \mathbb{R} \times \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{m \times n}$

(5) $\lambda \cdot (\mu \cdot A) = (\lambda \cdot \mu) \cdot A$

(6) $1 \cdot A = A$

(c) distributive laws:

(7) $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$

(8) $(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$

Linear Algebra – Part 12

Example: Xavier is two years older than Yasmin.
Together they are 40 years old.

How old is Xavier?
How old is Yasmin?

$$x = y + 2$$
$$x + y = 40$$

two unknowns and two equations

Another Example:

$$\left.\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= -7 \\ -3x_1 + x_2 - x_3 &= 0 \\ 20x_1 + 10x_2 &= 80 \\ 10x_2 + 25x_3 &= 90 \end{aligned}\right\}$$

4 equations and 3 unknowns x_1, x_2, x_3

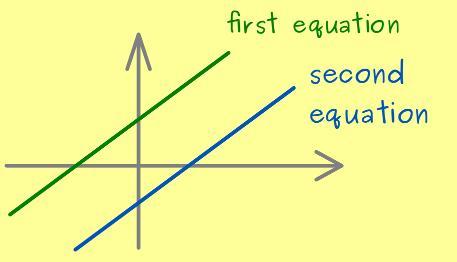
Linear equation: $\text{constant} \cdot x_1 + \text{constant} \cdot x_2 + \cdots + \text{constant} \cdot x_n = \text{constant}$

Definition: System of linear equations (LES) with m equations and n unknowns:

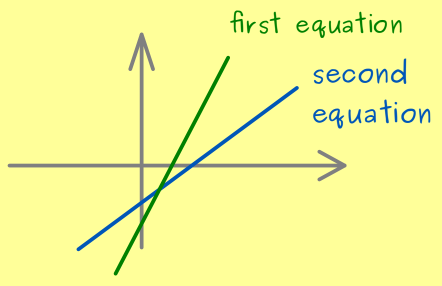
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A solution of the LES: choice of values for $x_1, \dots, x_n$ such that all m equations are satisfied.

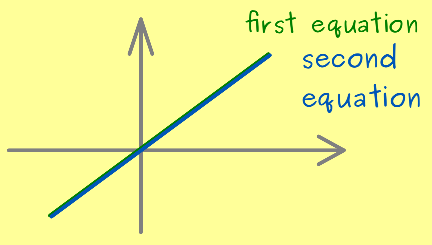
Note: – it's possible that there is no solution $m = 2$, $n = 2$



– it's possible that there is a unique solution $m = 2$, $n = 2$



– it's possible that there are infinitely many solutions



Short notation: Instead of

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

we write $Ax = b$ with $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

and $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Example:

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= -7 \\ -3x_1 + x_2 - x_3 &= 0 \\ 20x_1 + 10x_2 &= 80 \\ 10x_2 + 25x_3 &= 90 \end{aligned}$$

can be written as

$$\begin{pmatrix} 2 & -3 & 4 \\ -3 & 1 & -1 \\ 20 & 10 & 0 \\ 0 & 10 & 25 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 80 \\ 90 \end{pmatrix}$$

matrix-vector product

"matrix times vector = vector"

Linear Algebra – Part 13

Names for matrices: $A \in \mathbb{R}^{m \times n}$ — m: number of rows, n: number of columns

square matrix: $A \in \mathbb{R}^{n \times n}$ for example: $\begin{pmatrix} 1 & 7 & 9 \\ 2 & 8 & 2 \\ 4 & 1 & 3 \end{pmatrix}$

column vector: $A \in \mathbb{R}^{m \times 1}$ for example: $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

row vector: $A \in \mathbb{R}^{1 \times n}$ for example: $\begin{pmatrix} 2 & 4 & 6 & 7 \end{pmatrix}$

scalar: $A \in \mathbb{R}^{1 \times 1}$ for example: (4)

diagonal matrix: $A \in \mathbb{R}^{m \times n}$, $a_{ij} = 0$ for $i \neq j$

$$\begin{pmatrix} \blacksquare & 0 & 0 & 0 & 0 & 0 \\ 0 & \blacksquare & 0 & 0 & 0 & 0 \\ 0 & 0 & \blacksquare & 0 & 0 & 0 \\ 0 & 0 & 0 & \blacksquare & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{pmatrix}$$

upper triangular matrix: $A \in \mathbb{R}^{n \times n}$

$a_{ij} = 0$ for $i > j$

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \\ 0 & 0 & \blacksquare \end{pmatrix}$$

lower triangular matrix: $A \in \mathbb{R}^{n \times n}$

$a_{ij} = 0$ for $i < j$

$$\begin{pmatrix} \blacksquare & 0 & 0 & 0 \\ \blacksquare & \blacksquare & 0 & 0 \\ \blacksquare & \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{pmatrix}$$

symmetric matrix: $A \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 3 \\ 1 & 4 & 2 \\ 3 & 2 & -5 \end{pmatrix}$$

$a_{ij} = a_{ji}$ for all i,j

skew-symmetric matrix: $A \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ -\blacksquare & \blacksquare & \blacksquare \\ -\blacksquare & -\blacksquare & \blacksquare \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

$a_{ij} = -a_{ji}$ for all i,j

Linear Algebra – Part 14

Column picture: $A \in \mathbb{R}^{m \times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} , \quad a_i := \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

Matrix-vector product:

$$Ax = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1 \cdot \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + x_2 \cdot \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} + \cdots + x_n \cdot \begin{pmatrix} | \\ a_n \\ | \end{pmatrix}$$

vector $x \in \mathbb{R}^n \xrightarrow{\text{input}} \boxed{A\,\cdot} \xrightarrow{\text{output}} Ax \in \mathbb{R}^m$

Definition: $f_A : \mathbb{R}^n \to \mathbb{R}^m , \quad x \mapsto Ax$

linear map

Linear Algebra – Part 15

$A \in \mathbb{R}^{m \times n}$ ← collection of m row vectors

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{——} \; \alpha_1^T \; \text{——} \\ \text{——} \; \alpha_2^T \; \text{——} \\ \vdots \\ \text{——} \; \alpha_m^T \; \text{——} \end{pmatrix}$$

$$\alpha_i^T := \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}$$

T stands for "transpose"

flat matrix $\mathbb{R}^{1 \times n}$ → $u^T = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}^T = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$

transpose of column vector = row vector

$u^T x$ for $x \in \mathbb{R}^n$ is defined.

Example: $\begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 = \left\langle \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right\rangle$

standard inner product

Remember: For $u, v \in \mathbb{R}^n$: $u^T v = \langle u, v \rangle$

Row picture of the matrix-vector multiplication:

$$Ax = \begin{pmatrix} \text{——} \; \alpha_1^T \; \text{——} \\ \text{——} \; \alpha_2^T \; \text{——} \\ \vdots \\ \text{——} \; \alpha_m^T \; \text{——} \end{pmatrix} \underbrace{\begin{pmatrix} | \\ x \\ | \end{pmatrix}}_{\in \mathbb{R}^n} = \begin{pmatrix} \alpha_1^T x \\ \alpha_2^T x \\ \vdots \\ \alpha_m^T x \end{pmatrix} \in \mathbb{R}^m$$

Example:

$$\begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 1 \cdot 1 + 2 \cdot 0 \\ 3 \cdot 3 + 2 \cdot 1 + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$$

Linear Algebra – Part 16

matrix $\cdot$ matrix $=$ matrix (matrix product)

$$A \in \mathbb{R}^{m \times n}, \; b \in \mathbb{R}^{n} \rightsquigarrow Ab \in \mathbb{R}^{m}$$

$$A \in \mathbb{R}^{m \times n}, \; b_1, \dots, b_k \in \mathbb{R}^{n} \rightsquigarrow Ab_1, Ab_2, \dots, Ab_k \in \mathbb{R}^{m}$$

$$\underbrace{A}_{\in \mathbb{R}^{m \times n}} \cdot \underbrace{\begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{pmatrix}}_{\in \mathbb{R}^{n \times k}} := \underbrace{\begin{pmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_k \\ | & | & & | \end{pmatrix}}_{\in \mathbb{R}^{m \times k}}$$

Definition: For $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times k}$, define the matrix product AB:

$$AB = \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ \vdots \\ — \alpha_m^T — \end{pmatrix} \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \alpha_1^T b_1 & \alpha_1^T b_2 & \cdots & \alpha_1^T b_k \\ \alpha_2^T b_1 & \alpha_2^T b_2 & \cdots & \alpha_2^T b_k \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m^T b_1 & \alpha_m^T b_2 & \cdots & \alpha_m^T b_k \end{pmatrix}$$

Example:

$$\begin{array}{cc} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} & \boxed{\begin{matrix} 4 & 5 \\ 10 & 11 \end{matrix}} \end{array} \implies AB = \begin{pmatrix} 4 & 5 \\ 10 & 11 \end{pmatrix}$$

Linear Algebra – Part 17

matrix product: $\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times k} \longrightarrow \mathbb{R}^{m \times k}$

$$(A, B) \longmapsto AB$$

defined by: $$(AB)_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}$$

Properties: (a) $$(A+B)C = AC + BC$$
$$D(A+B) = DA + DB$$ (distributive laws)

(b) $$\lambda \cdot (AB) = (\lambda \cdot A)B = A(\lambda \cdot B)$$

(c) $$(AB)C = A(BC)$$ (associative law)

Proof: (c) $$((AB)C)_{ij} = \sum_{\ell=1}^{n} (AB)_{i\ell} c_{\ell j}$$
$$= \sum_{\ell} \Big(\sum_{z} a_{iz} b_{z\ell} \Big) c_{\ell j}$$
$$= \sum_{z} a_{iz} \sum_{\ell} b_{z\ell} c_{\ell j} = \sum_{z} a_{iz} (BC)_{zj}$$
$$= (A(BC))_{ij}$$

Important: no commutative law (in general)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$
$$\neq$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Linear Algebra – Part 18

linear = conserves structure of a vector space

For the vector space $\mathbb{R}^n$: vector addition + scalar multiplication $\lambda \cdot$

Definition: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called linear if for all $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$:

(a) $f(\underbrace{x + y}_{\text{addition in } \mathbb{R}^n}) = f(x) \underbrace{+}_{\text{addition in } \mathbb{R}^m} f(y)$

(b) $f(\lambda \cdot x) = \lambda \cdot f(x)$

Example:

(1) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x$ linear

(2) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ not linear because $f(3 \cdot 1) = 9$, $3 \cdot f(1) = 3$ #

(3) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x + 1$ not linear because $f(0 \cdot 1) = 1$, $0 \cdot f(1) = 0$ #

Linear Algebra – Part 19

$$A \in \mathbb{R}^{m \times n} \rightsquigarrow f_A : \mathbb{R}^n \to \mathbb{R}^m$$
$$x \mapsto Ax$$

Proposition: f_A is a linear map:

(1) $f_A(x+y) = f_A(x) + f_A(y)$, $A(x+y) = Ax + Ay$ (distributive)

(2) $f_A(\lambda \cdot x) = \lambda \cdot f_A(x)$, $A(\lambda \cdot x) = \lambda \cdot (Ax)$ (compatible)

Example:
$$\begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$
$$= \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix}(x_1 + y_1) + \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix}(x_2 + y_2)$$
$$= \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} x_1 + \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} x_2 + \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} y_1 + \begin{pmatrix} | \\ a_2 \\ | \end{pmatrix} y_2$$
$$= \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

matrix A (table of numbers) $\leftrightsquigarrow$ f_A abstract linear map

Now: two matrices A, B

$$\left.\begin{matrix} A \in \mathbb{R}^{m \times k} \\ B \in \mathbb{R}^{k \times n} \end{matrix}\right\} AB \in \mathbb{R}^{m \times n} \leftrightsquigarrow$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "f_B"] \arrow[rr, bend right, "f_A \circ f_B"'] & \mathbb{R}^k \arrow[r, bend left, "f_A"] & \mathbb{R}^m \end{tikzcd}$$

$$(\underbrace{f_A \circ f_B}_{f_{AB}})(x) = f_A(f_B(x)) = f_A(Bx) = A(Bx) = (AB)x$$

Linear Algebra – Part 20

Linear map: $f: \mathbb{R}^n \to \mathbb{R}^m, \quad x \mapsto f(x)$

n components

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

canonical unit vectors

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \overset{\text{linearity}}{=} x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n)$$

$\Rightarrow$ to know $f(x)$, it's sufficient to know $f(e_1), \dots, f(e_n)$

Proposition: $f: \mathbb{R}^n \to \mathbb{R}^m$ linear.

Then there is exactly one matrix $A \in \mathbb{R}^{m \times n}$ with $f = f_A$ $\big(f(x) = Ax \big)$

and

$$A = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}.$$

Proof: $f_A(x) = f_A\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$= \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ f(e_1) \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ f(e_n) \\ | \end{pmatrix}$$

$$= f(x)$$

Uniqueness: Assume there are $A, B \in \mathbb{R}^{m \times n}$ with $f = f_A$ and $f = f_B$

$\Rightarrow \quad Ax = Bx$ for all $x \in \mathbb{R}^n$

$\Rightarrow \quad (A - B)x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ for all $x \in \mathbb{R}^n$

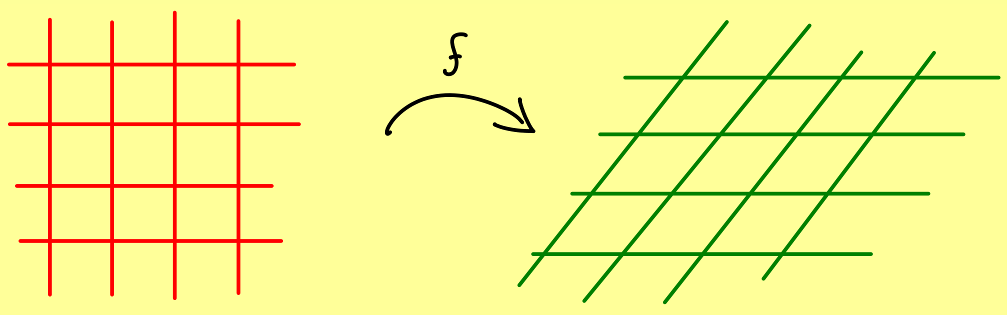
$\overset{\text{Use } e_i}{\Rightarrow} \quad A - B = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \Rightarrow A = B$ $\square$

The Bright Side of Mathematics

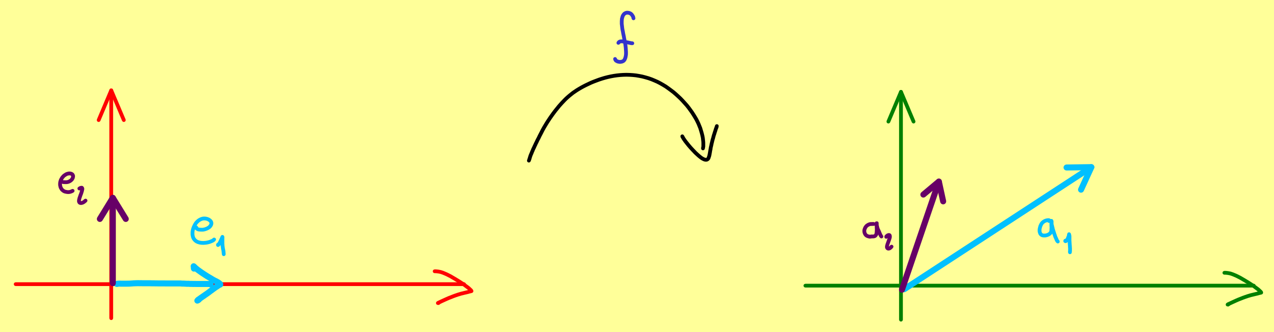
Linear Algebra – Part 21

$f: \mathbb{R}^n \to \mathbb{R}^m$ linear

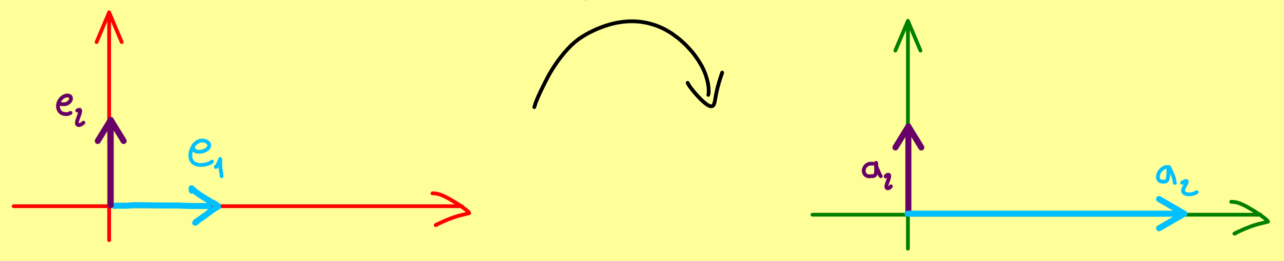
- preserves the linear structure
- linear subspaces are sent to linear subspaces



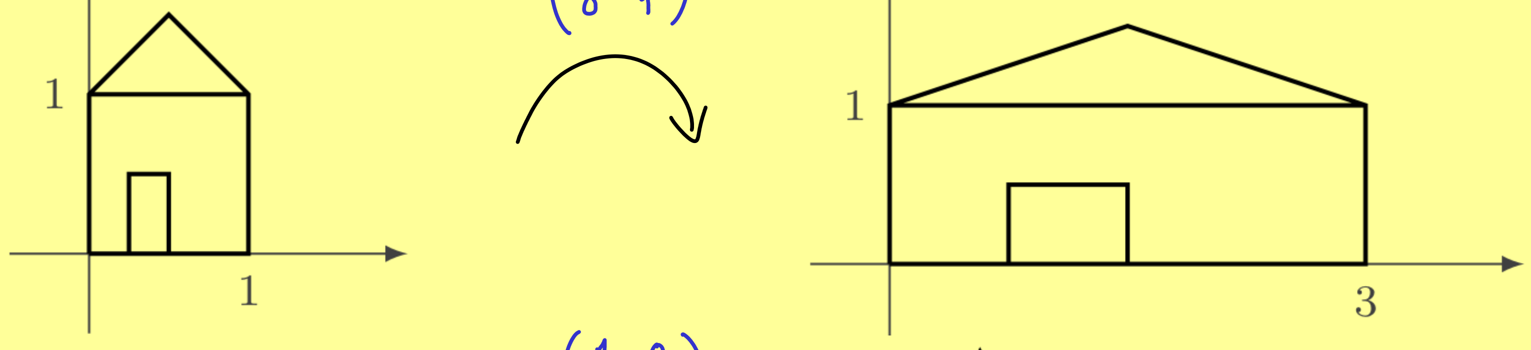
<u>Examples:</u> $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix} x$



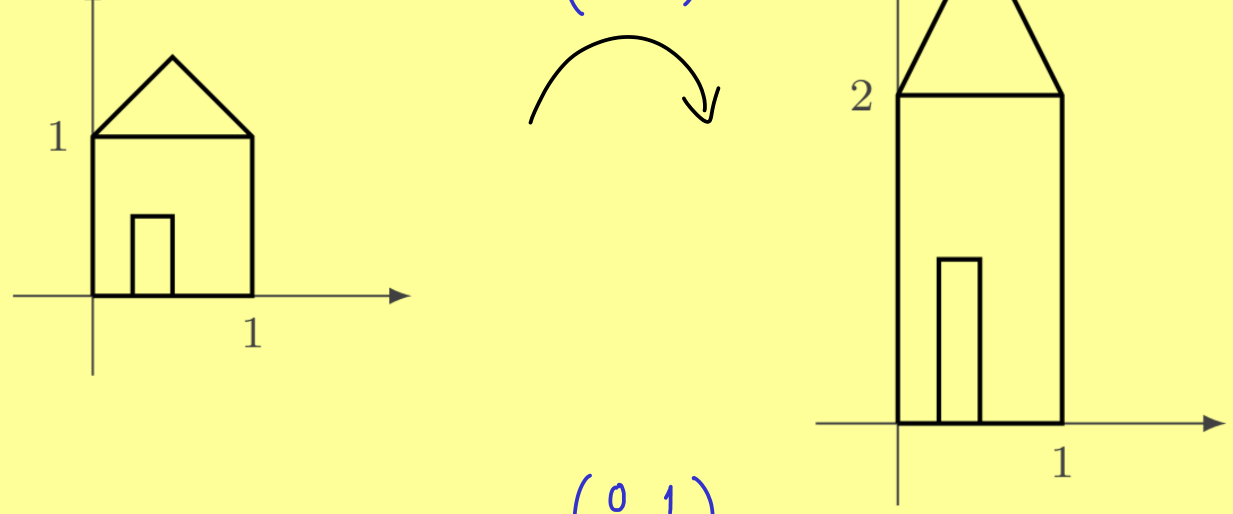
$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$



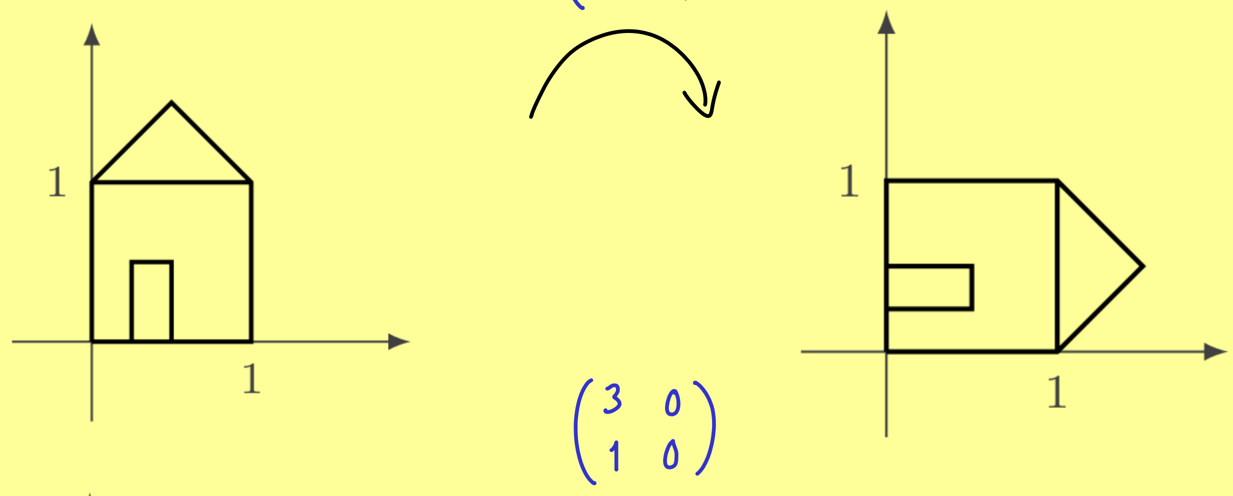
$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$



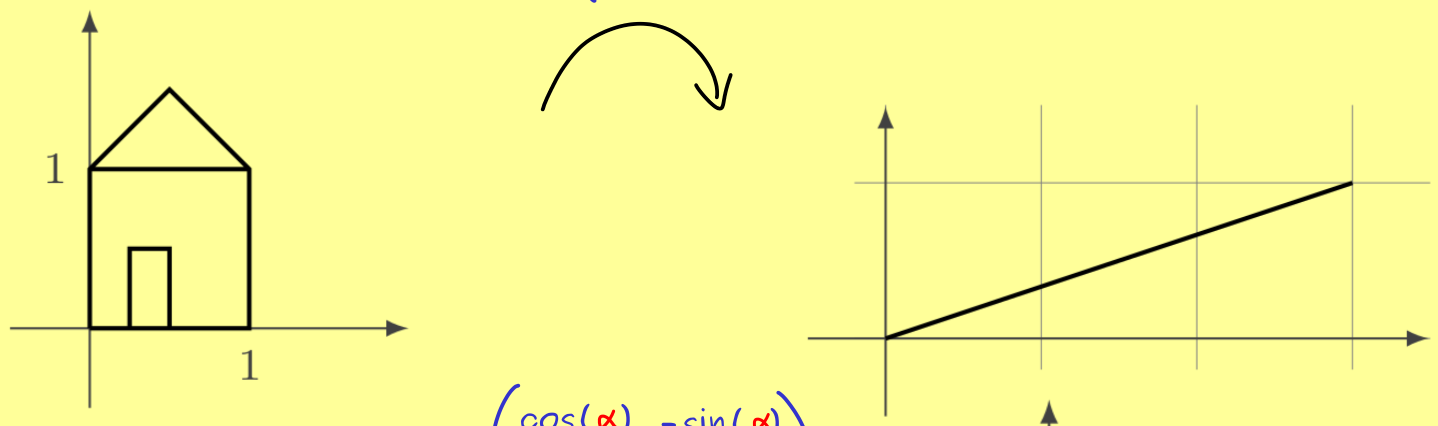
$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$



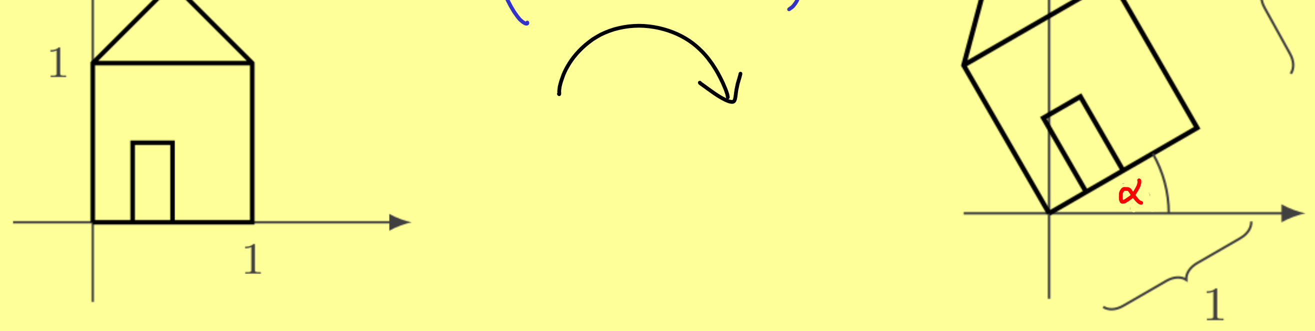
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$\begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$

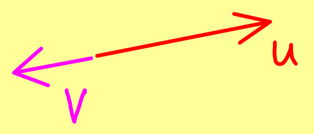


$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$



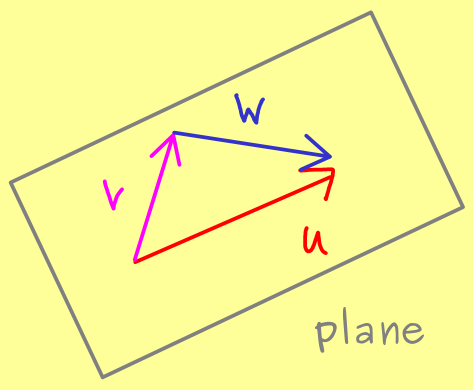
Linear Algebra – Part 22

$\mathbb{R}^2$:



colinear: $u = \lambda v$

$\mathbb{R}^3$:



coplanar: $u = \lambda v + \mu w$

$\iff 0 = (-1) u + \lambda v + \mu w$

Definition: Let $v^{(1)}, v^{(2)}, \dots, v^{(k)} \in \mathbb{R}^n$. The family $\left(v^{(1)}, v^{(2)}, \dots, v^{(k)}\right)$ (or $\{v^{(1)}, v^{(2)}, \dots, v^{(k)}\}$) is called linearly dependent if there are $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$ that are not all equal to zero such that:

$$\sum_{j=1}^{k} \lambda_j v^{(j)} = 0$$

zero vector in $\mathbb{R}^n$

We call the family linearly independent if

$$\sum_{j=1}^{k} \lambda_j v^{(j)} = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \dots = 0$$

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Linear Algebra – Part 23

$(v^{(1)}, v^{(2)}, \dots, v^{(k)})$ linearly independent if

$$\sum_{j=1}^{k} \lambda_j v^{(j)} = 0 \implies \lambda_1 = \lambda_2 = \lambda_3 = \dots = 0$$

Examples: (a) $(v^{(1)})$ linearly independent if $v^{(1)} \neq 0$

(b) $(0, v^{(2)}, \dots, v^{(k)})$ linearly dependent

$$(\lambda_1 = 1 \quad , \lambda_2 = \lambda_3 = \dots = 0)$$

(c) $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ linearly dependent

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

(d) $(e_1, e_2, \dots, e_n)$, $e_i \in \mathbb{R}^n$ canonical unit vectors

linearly independent

$$\sum_{j=1}^{n} \lambda_j e_j = 0 \iff \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \iff \lambda_1 = \lambda_2 = \lambda_3 = \dots = 0$$

(e) $(e_1, e_2, \dots, e_n, v)$, $e_i, v \in \mathbb{R}^n$

linearly dependent

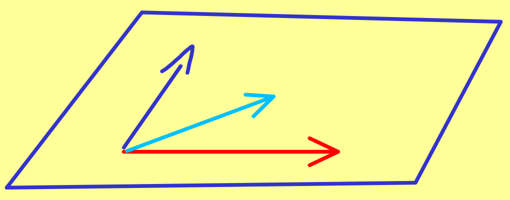
Fact: $(v^{(1)}, v^{(2)}, \dots, v^{(k)})$ family of vectors $v^{(j)} \in \mathbb{R}^n$

linearly dependent

$\iff$ There is ℓ with

$$\mathrm{Span}(v^{(1)}, v^{(2)}, \dots, v^{(k)}) = \mathrm{Span}(v^{(1)}, \dots, v^{(\ell-1)}, v^{(\ell+1)}, \dots, v^{(k)})$$

Linear Algebra – Part 24

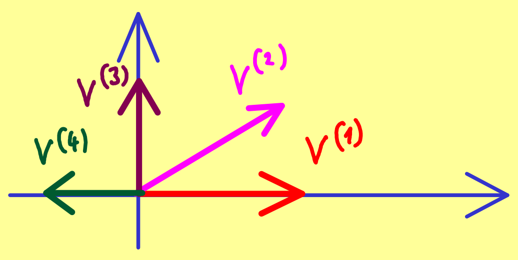
subspace:  $U \subseteq \mathbb{R}^n$ with

(a) $0 \in U$

(b) $u \in U, \lambda \in \mathbb{R} \Rightarrow \lambda \cdot u \in U$

(c) $u, v \in U \Rightarrow u + v \in U$

plane: $\mathbb{R}^2$



$$\mathrm{Span}(v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}) = \mathbb{R}^2$$

$$\mathrm{Span}(v^{(1)}, v^{(3)}) = \mathbb{R}^2$$

$$\mathrm{Span}(v^{(1)}, v^{(4)}) = \mathbb{R} \times \{0\} \neq \mathbb{R}^2$$

Definition: $U \subseteq \mathbb{R}^n$ subspace, $\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$, $v^{(j)} \in \mathbb{R}^n$.

$\mathcal{B}$ is called a basis of U if:

(a) $U = \mathrm{Span}(\mathcal{B})$

(b) $\mathcal{B}$ is linearly independent

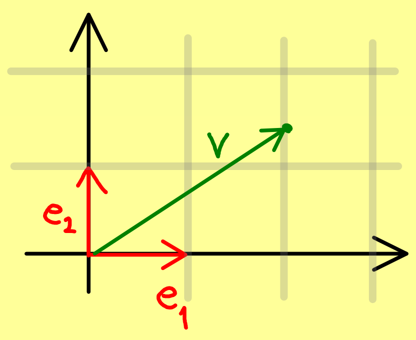
Example: $\mathbb{R}^n = \mathrm{Span}(\underbrace{e_1, \dots, e_n}_{\text{standard basis of } \mathbb{R}^n})$

$$\mathbb{R}^3 = \mathrm{Span}\Bigg(\underbrace{\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\text{basis of } \mathbb{R}^3}\Bigg)$$

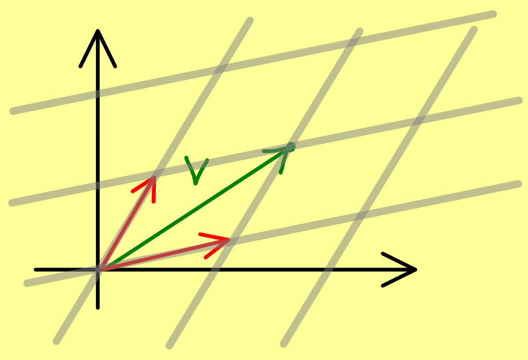
Linear Algebra – Part 25

basis of a subspace: spans the subspace + linearly independent

$\mathbb{R}^2$



$v = \begin{pmatrix} 2 \\ \frac{4}{3} \end{pmatrix}$



coordinates of v: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

coordinates: $U \subseteq \mathbb{R}^n$ subspace, $\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ basis of U

$\Longrightarrow$ Each vector $u \in U$ can be written as a linear combination:

$$u = \lambda_1 v^{(1)} + \lambda_2 v^{(2)} + \dots + \lambda_k v^{(k)} \quad , \quad \lambda_j \in \mathbb{R}$$

(uniquely determined)

coordinates of u with respect to $\mathcal{B}$

$$u = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}_{\mathcal{B}}$$

Example: $\mathbb{R}^3 = \mathrm{Span}\Big(\underbrace{\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{\text{basis of } \mathbb{R}^3}\Big)$

$$u = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$\tilde{u} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = -1 \cdot \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

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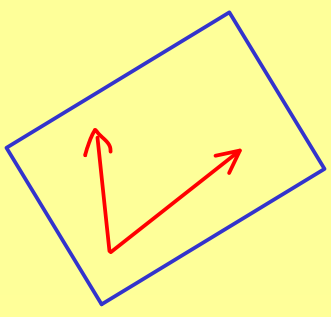
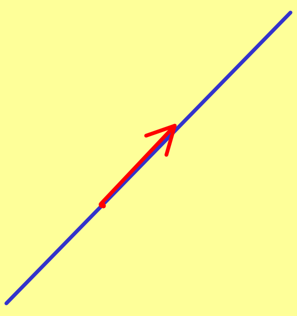
Linear Algebra – Part 26



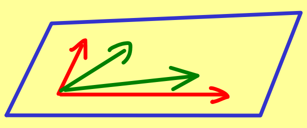
dimension = 2



dimension = 1

Steinitz Exchange Lemma

Let $U \subseteq \mathbb{R}^n$ be a subspace and



$\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ be a basis of U.

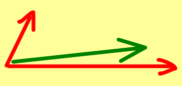
$\mathcal{A} = (a^{(1)}, a^{(2)}, \dots, a^{(\ell)})$ linearly independent vectors in U.

Then: One can add $k-\ell$ vectors from $\mathcal{B}$ to the family $\mathcal{A}$ such that we get a new basis of U.

Proof: $\ell = 1$: $\mathcal{B} \cup \mathcal{A} = (v^{(1)}, v^{(2)}, \dots, v^{(k)}, a^{(1)})$ is linearly dependent

because $\mathcal{B}$ is a basis: there are uniquely given $\lambda_1, \dots, \lambda_k \in \mathbb{R}$:

$$(*) \qquad a^{(1)} = \lambda_1 v^{(1)} + \dots + \lambda_k v^{(k)}$$



Choose $\lambda_j \neq 0$:

$$v^{(j)} = \frac{1}{-\lambda_j}\left(\lambda_1 v^{(1)} + \dots + \lambda_{j-1} v^{(j-1)} + \lambda_{j+1} v^{(j+1)} + \dots + \lambda_k v^{(k)} - a^{(1)}\right)$$

Remove $v^{(j)}$ from $\mathcal{B} \cup \mathcal{A}$ and call it $\mathcal{C}$.

$\mathcal{C}$ is linearly independent:

$$\tilde{\lambda}_1 v^{(1)} + \dots + \tilde{\lambda}_{j-1} v^{(j-1)} + \tilde{\lambda}_j a^{(1)} + \tilde{\lambda}_{j+1} v^{(j+1)} + \dots + \tilde{\lambda}_k v^{(k)} = 0$$

Assume $\tilde{\lambda}_j \neq 0$: $a^{(1)} =$ linear combination with $v^{(1)}, \dots, v^{(j-1)}, v^{(j+1)}, \dots, v^{(k)}$ $\lightning$ $(*)$

Hence: $\tilde{\lambda}_j = 0 \Rightarrow$

$$\tilde{\lambda}_1 v^{(1)} + \dots + \tilde{\lambda}_{j-1} v^{(j-1)} + \tilde{\lambda}_{j+1} v^{(j+1)} + \dots + \tilde{\lambda}_k v^{(k)} = 0$$

$$\xRightarrow{\text{lin. independence}} \tilde{\lambda}_i = 0 \quad \text{for } i \in \{1, \dots, k\}$$

$\mathcal{C}$ spans U : $u \in U \overset{\mathcal{B}\text{ basis}}{\Rightarrow}$ there are coefficients

$$\boxed{v^{(j)} = \frac{1}{-\lambda_j}\left(\lambda_1 v^{(1)} + \dots + \lambda_{j-1} v^{(j-1)} + \lambda_{j+1} v^{(j+1)} + \dots + \lambda_k v^{(k)} - a^{(1)}\right)}$$

$$u = \mu_1 v^{(1)} + \dots + \mu_{j-1} v^{(j-1)} + \mu_j v^{(j)} + \mu_{j+1} v^{(j+1)} + \dots + \mu_k v^{(k)}$$

$$= \tilde{\mu}_1 v^{(1)} + \dots + \tilde{\mu}_{j-1} v^{(j-1)} + \tilde{\mu}_j a^{(1)} + \tilde{\mu}_{j+1} v^{(j+1)} + \dots + \tilde{\mu}_k v^{(k)}$$

Linear Algebra – Part 27

Steinitz Exchange Lemma: $(v^{(1)}, v^{(2)}, \dots, v^{(k)})$ basis of U

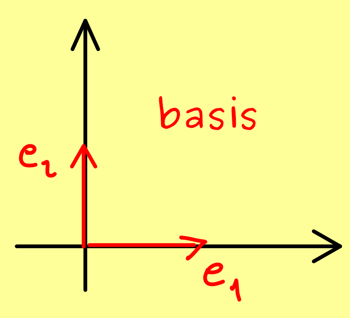
$(a^{(1)}, a^{(2)}, \dots, a^{(\ell)})$ lin. independent vectors in U

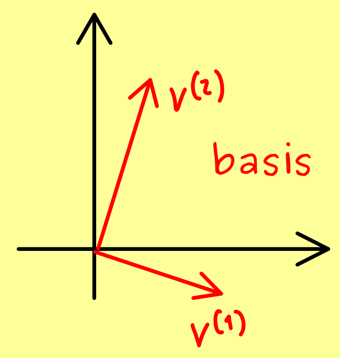
$\Rightarrow$ new basis of U

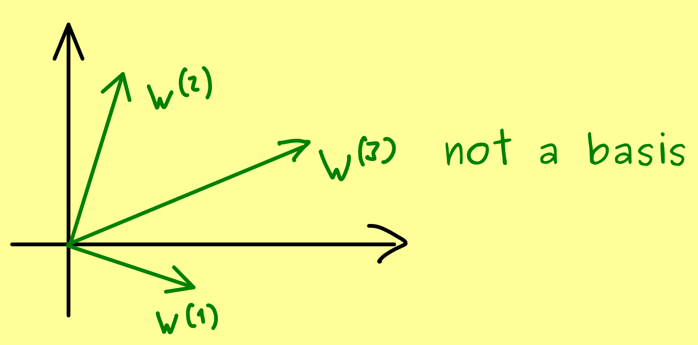
Fact: Let $U \subseteq \mathbb{R}^n$ be a subspace and $\mathcal{B} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ be a basis of U.

Then: (a) Each family $(w^{(1)}, w^{(2)}, \dots, w^{(m)})$ with $m > k$ vectors in U is linearly dependent.

(b) Each basis of U has exactly k elements.







Definition: Let $U \subseteq \mathbb{R}^n$ be a subspace and $\mathcal{B}$ be a basis of U.

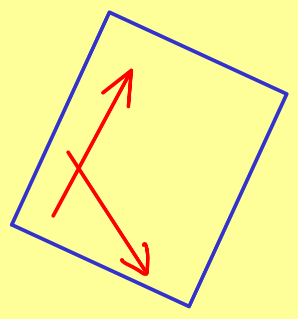
The number of vectors in $\mathcal{B}$ is called the dimension of U.

We write: $\dim(U)$ ← integer

set: $\dim(\{0\}) := 0$ $\quad\quad (\operatorname{Span}(\emptyset) = \{0\})$ ← basis

Example: $(e_1, e_2, \dots, e_n)$ standard basis of $\mathbb{R}^n$

$$\dim(\mathbb{R}^n) = n$$



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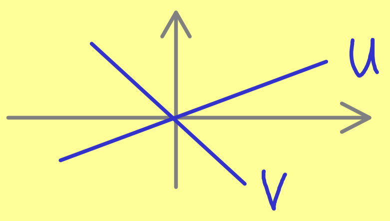
Linear Algebra – Part 28

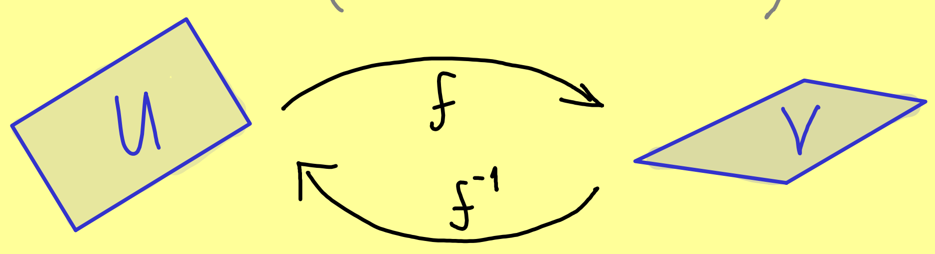
Dimension of U: number of elements in a basis of U $= \dim(U)$

Theorem: $U, V \subseteq \mathbb{R}^n$ linear subspaces



(a) $\dim(U) = \dim(V) \iff$ there is a bijective linear map $f: U \to V$

$\left(f^{-1}: V \to U \text{ linear} \right)$



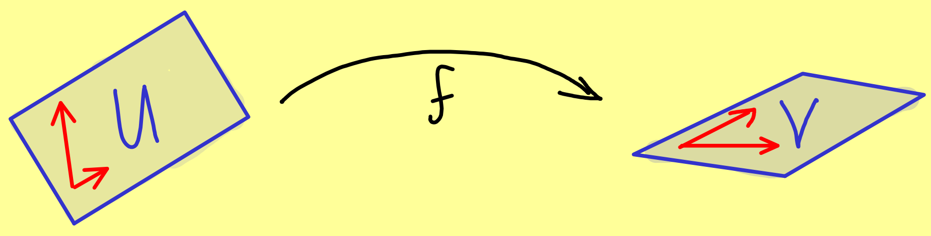
(b) $U \subseteq V$ and $\dim(U) = \dim(V) \implies U = V$

Proof: (a) ($\Rightarrow$) We assume $\dim(U) = \dim(V)$.

Hence: $\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ basis of U

$\mathcal{C} = (v^{(1)}, v^{(2)}, \dots, v^{(k)})$ basis of V

define: $f: U \to V$, $f(u^{(i)}) = v^{(i)}$



For $x \in U$: $f(x) = f\left(\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)}\right)$ uniquely determined $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

$= \lambda_1 \cdot f(u^{(1)}) + \lambda_2 \cdot f(u^{(2)}) + \dots + \lambda_k \cdot f(u^{(k)})$

$= \lambda_1 \cdot v^{(1)} + \dots + \lambda_k \cdot v^{(k)} =: f(x)$

Now define: $f^{-1}: V \to U$, $f^{-1}(v^{(i)}) = u^{(i)}$

Then: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y \implies$ f is bijective+linear

($\Leftarrow$) We assume that there is $\underbrace{\text{bijective}}_{\text{injective+surjective}}$ linear map $f: U \to V$.

Let $\mathcal{B} = (u^{(1)}, u^{(2)}, \dots, u^{(k)})$ be a basis of U

$\implies \left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)}) \right)$ basis in V ?

f injective: linearly independent

f surjective: $\operatorname{Span}\left(f(u^{(1)}), f(u^{(2)}), \dots, f(u^{(k)})\right) = V$

$\implies \dim(U) = \dim(V)$

(b) We show: $\boxed{U \subseteq V \text{ and } \dim(U) = \dim(V) \implies U = V}$

$(u^{(1)}, u^{(2)}, \dots, u^{(k)})$ basis of U $\implies$ $(u^{(1)}, u^{(2)}, \dots, u^{(k)})$ basis of V

$v = \lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \dots + \lambda_k u^{(k)}$

$\in U$

$\implies U = V$ $\square$

Linear Algebra – Part 29

$A \in \mathbb{R}^{m \times n} \longleftrightarrow f_A : \mathbb{R}^n \to \mathbb{R}^m$ linear map

Definition: Identity matrix in $\mathbb{R}^{n \times n}$:

$$\mathbb{1}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

other notations: I_n, id, Id, E_n

Properties:

$\mathbb{1}_n B = B$ for $B \in \mathbb{R}^{n \times m}$

$A \cdot \mathbb{1}_n = A$ for $A \in \mathbb{R}^{m \times n}$

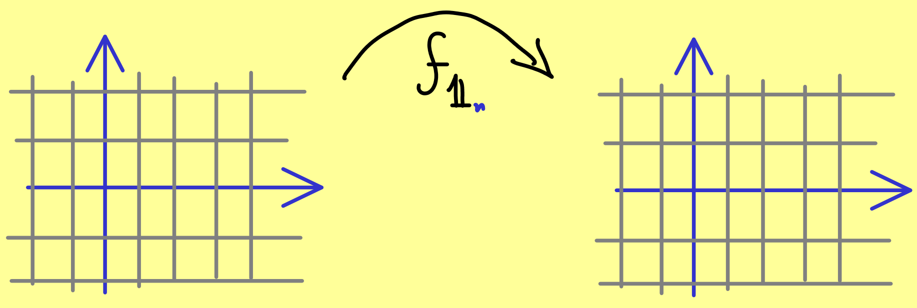
neutral element with respect to the matrix multiplication

Map level:

$$f_{\mathbb{1}_n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$x \longmapsto \underbrace{\mathbb{1}_n x}_{= x}$$

$f_{\mathbb{1}_n}$ = identity map



Inverses: $A \in \mathbb{R}^{n \times n} \rightsquigarrow \tilde{A} \in \mathbb{R}^{n \times n}$ with $A\tilde{A} = \mathbb{1}$ and $\tilde{A}A = \mathbb{1}$

If such a $\tilde{A}$ exists, it's uniquely determined. Write A^{-1} (instead of $\tilde{A}$)

inverse of A

Definition: A matrix $A \in \mathbb{R}^{n \times n}$ is called invertible (= non-singular = regular) if the corresponding linear map $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is bijective.

Otherwise we call A singular.

A matrix $\tilde{A} \in \mathbb{R}^{n \times n}$ is called the inverse of A if $f_{\tilde{A}} = (f_A)^{-1}$

Write A^{-1} (instead of $\tilde{A}$)

Summary:

$$\begin{aligned} f_{A^{-1}} \circ f_A &= id \\ f_A \circ f_{A^{-1}} &= id \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} A^{-1}A &= \mathbb{1} \\ AA^{-1} &= \mathbb{1} \end{aligned}$$

Linear Algebra – Part 30

injectivity, surjectivity, bijectivity for square matrices

system of linear equations: $Ax = b \overset{\text{if } A \text{ invertible}}{\Longrightarrow} A^{-1}Ax = A^{-1}b \Longrightarrow x = A^{-1}b$

Theorem: $A \in \mathbb{R}^{n \times n}$ square matrix. $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induced linear map.

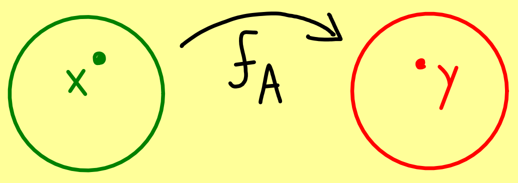
Then: f_A is injective $\Longleftrightarrow$ f_A is surjective

Proof: $(\Rightarrow)$ f_A injective, standard basis of $\mathbb{R}^n$ $(e_1, \dots, e_n)$

$\Longrightarrow \underbrace{(f_A(e_1), \dots, f_A(e_n))}_{\text{basis of } \mathbb{R}^n}$ still linearly independent

$\Longrightarrow f_A$ is surjective

$(\Leftarrow)$ f_A surjective



For each $y \in \mathbb{R}^n$, you find $x \in \mathbb{R}^n$ with $f_A(x) = y$.

We know: $x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$

$y = f_A(x) = x_1 f_A(e_1) + x_2 f_A(e_2) + \dots + x_n f_A(e_n)$

$\Longrightarrow (f_A(e_1), \dots, f_A(e_n))$ spans $\mathbb{R}^n$

$\overset{n \text{ vectors}}{\Longrightarrow} (f_A(e_1), \dots, f_A(e_n))$ linearly independent

Assume $f_A(x) = f_A(\tilde{x}) \Longrightarrow f_A(\underbrace{x - \tilde{x}}_{v}) = 0$

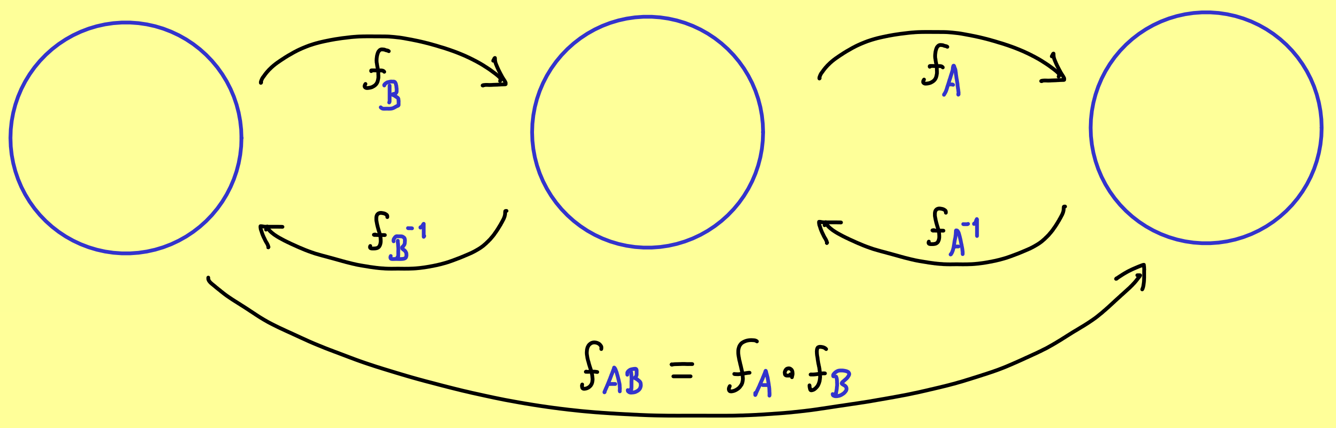
$\Longrightarrow v_1 f_A(e_1) + v_2 f_A(e_2) + \dots + v_n f_A(e_n) = 0$

$\overset{\text{lin. independence}}{\Longrightarrow} v_1 = v_2 = \dots = v_n = 0$

$\Longrightarrow x = \tilde{x} \Longrightarrow f_A$ is injective $\square$

Linear Algebra – Part 31

matrices $A, B \in \mathbb{R}^{n \times n}$



We have: $f_{B^{-1}} \circ f_{A^{-1}} = (f_{AB})^{-1} \Rightarrow \boxed{(AB)^{-1} = B^{-1} A^{-1}}$

Important fact: $f: \mathbb{R}^n \to \mathbb{R}^n$ linear and bijective

$\Rightarrow f^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is also linear

Proof:

$$f^{-1}(\lambda y) = f^{-1}(\lambda \cdot f(x)) \underbrace{=}_{f \text{ linear}} f^{-1}(f(\lambda x)) = \lambda \cdot x = \lambda f^{-1}(y) \checkmark$$

There is exactly one x with $f(x) = y$

$$f^{-1}(y + \tilde{y}) = f^{-1}(f(x) + f(\tilde{x})) \underbrace{=}_{f \text{ linear}} f^{-1}(f(x + \tilde{x})) = x + \tilde{x}$$

$$= f^{-1}(y) + f^{-1}(\tilde{y}) \checkmark$$

Linear Algebra – Part 32

Transposition: changing the roles of columns and rows

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

For $a \in \mathbb{R}^n$ we have: $(a^T)^T = a$

Definition: For $A \in \mathbb{R}^{m \times n}$ we define $A^T \in \mathbb{R}^{n \times m}$ (transpose of A) by:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Examples:

(a) $$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}$$

(b) $$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

(c) $$A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 0 \\ 5 & 0 & 3 \end{pmatrix}$$ (symmetric matrix)

Remember: $(AB)^T = B^T A^T$

Linear Algebra – Part 33

$A \in \mathbb{R}^{m \times n} \rightsquigarrow A^T \in \mathbb{R}^{n \times m}$

standard inner product in $\mathbb{R}^n \rightsquigarrow \langle u, v \rangle \in \mathbb{R}$
$= u^T v$

Proposition: For $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\underbrace{\langle y, Ax \rangle}_{\text{inner product in } \mathbb{R}^m} = \underbrace{\langle A^T y, x \rangle}_{\text{inner product in } \mathbb{R}^n}$$

Proof: $\langle \tilde{u}, \tilde{v} \rangle = \tilde{u}^T \tilde{v}$ for $\tilde{u}, \tilde{v} \in \mathbb{R}^m$

$(A^T y)^T = y^T \cdot (A^T)^T$

$$\langle y, \overbrace{Ax} \rangle = y^T (Ax) = (y^T A) x = (A^T y)^T x = \langle A^T y, x \rangle \qquad \square$$

Alternative definition: A^T is the only matrix $B \in \mathbb{R}^{n \times m}$ that satisfies:

$$\langle y, Ax \rangle = \langle By, x \rangle \quad \text{for all } x, y$$

Linear Algebra – Part 34

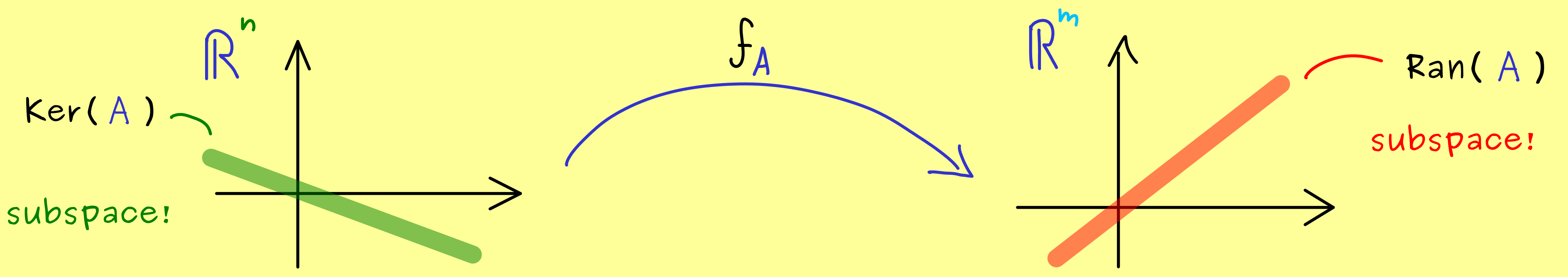
$A \in \mathbb{R}^{m \times n}$ induces a linear map $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \; x \longmapsto Ax$

$\mathrm{Ran}(A) := \{ Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$ range of A (image of A)

$= \mathrm{Ran}(f_A)$ (see Start Learning Sets – Part 5)

$\mathrm{Ker}(A) := \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subseteq \mathbb{R}^n$ kernel of A (nullspace of A)

$= f_A^{-1}[\{0\}]$ preimage of $\{0\}$ under f_A



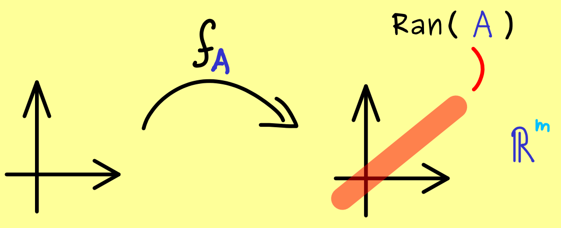
Remember: $\mathrm{Ran}(A) = \mathrm{Span}(a_1, a_2, \dots, a_n)$ $\qquad A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$

Solving LES? $Ax = b$ existence of solutions: $b \in \mathrm{Ran}(A)$?

uniqueness of solutions: $\mathrm{Ker}(A) \neq \{0\}$?

Linear Algebra – Part 35

Definition: For $A \in \mathbb{R}^{m \times n}$ we define:



$$\text{rank}(A) := \dim(\text{Ran}(A))$$

$$= \dim(\text{Span of columns of } A)$$

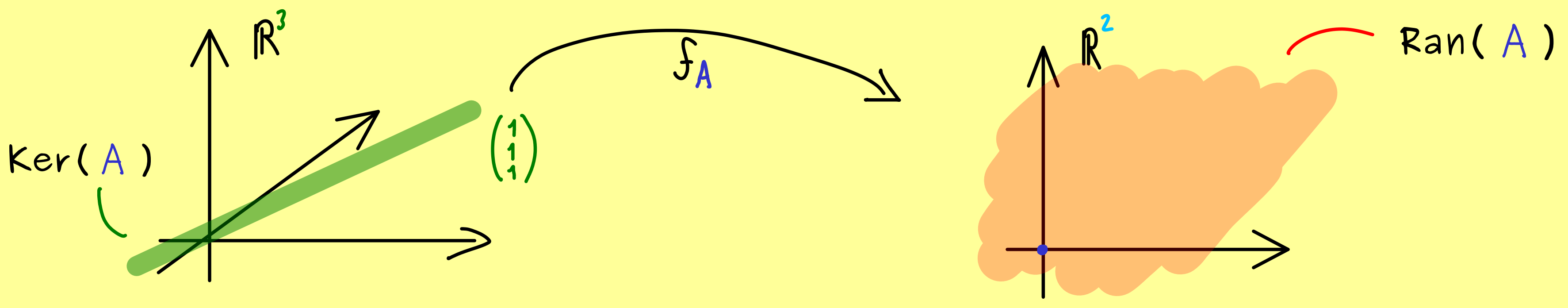
$$\leq \min(n, m)$$

A has full rank if $\text{rank}(A) = \min(n, m)$

Example: (a) $A = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}$, $\text{rank}(A) = 1$ (full rank)

(b) $A = \begin{pmatrix} 2 & 2 & -4 \\ 1 & 0 & -1 \end{pmatrix}$, $\text{rank}(A) = 2$ (full rank)

linearly independent



Definition: For $A \in \mathbb{R}^{m \times n}$ we define:

$$\text{nullity}(A) := \dim(\text{Ker}(A))$$

> **Rank-nullity theorem:** For $A \in \mathbb{R}^{m \times n}$ (n columns)
>
> $$\dim(\text{Ker}(A)) + \dim(\text{Ran}(A)) = n$$

Proof: $k = \dim(\text{Ker}(A))$. Choose: $(b_1, \dots, b_k)$ basis of $\text{Ker}(A)$.

Steinitz Exchange Lemma $\Rightarrow$ $(b_1, \dots, b_k, c_1, \dots, c_r)$ basis of $\mathbb{R}^n$

$r := n - k$

$$\text{Ran}(A) = \text{Span}(\underbrace{Ab_1}_{=0}, \dots, \underbrace{Ab_k}_{=0}, Ac_1, \dots, Ac_r)$$

$$= \text{Span}(Ac_1, \dots, Ac_r) \quad \Rightarrow \quad \dim(\text{Ran}(A)) \leq r$$

To show: $(Ac_1, \dots, Ac_r)$ is linearly independent

$$\lambda_1 Ac_1 + \lambda_2 Ac_2 + \cdots + \lambda_r Ac_r = 0$$

$$\overset{\text{linearity}}{=} A\Big(\sum_{i=1}^{r} \lambda_i c_i\Big) \quad \Rightarrow \quad \sum_{i=1}^{r} \lambda_i c_i \in \text{Ker}(A)$$

$$\overset{\text{basis of kernel}}{\Rightarrow} \quad \sum_{i=1}^{r} \lambda_i c_i = \sum_{j=1}^{k} \mu_j b_j \quad \Rightarrow \quad \sum_{i=1}^{r} \lambda_i c_i + \sum_{j=1}^{k} (-\mu_j) b_j = 0$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$$

$$\Rightarrow \quad \dim(\text{Ran}(A)) = r \qquad \square$$

Linear Algebra – Part 36

System of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 1 \\ 4x_1 + 6x_2 + 9x_3 &= 1 \\ 2x_1 + 4x_2 + 6x_3 &= 1 \end{aligned}$$

3 equations
3 unknowns

Short notation: $Ax = b$ $\xrightarrow{\text{augmented matrix}}$ $(A \mid b)$

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 6 & 9 & 1 \\ 2 & 4 & 6 & 1 \end{array}\right)$$

Example:

$x_1 + 3x_2 = 7$ (equation 1)

$2x_1 - x_2 = 0$ (equation 2) $\rightsquigarrow$ $x_2 = 2x_1$

put in equation 1

$\Rightarrow$ $x_1 + 3(2x_1) = 7$

$\Leftrightarrow$ $7x_1 = 7$ $\Leftrightarrow$ $x_1 = 1$ $\rightsquigarrow$ $x_2 = 2$

$\Rightarrow$ Only possible solution: $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Check? ✓

$\Rightarrow$ The system has a unique solution given by $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Better method: Gaussian elimination

Example:

$x_1 + 3x_2 = 7$ (equation 1)

$2x_1 - x_2 = 0$ (equation 2) $- 2 \cdot$ (equation 1)

eliminate x_1

$\rightsquigarrow$

$x_1 + 3x_2 = 7$ (equation 1)

$0 - 7x_2 = -14$ (equation 2)$' \cdot (-\frac{1}{7})$

$\rightsquigarrow$

$x_1 + 3x_2 = 7$ (equation 1)

$x_2 = 2$ (equation 2)$''$

$\Rightarrow$ $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ solution

Linear Algebra – Part 37

$$Ax = b \overset{\text{augmented matrix}}{\rightsquigarrow} (A \mid b)$$

$$A \overset{\text{reversible manipulation}}{\leftrightsquigarrow} \tilde{A} \; : \quad \underset{\text{invertible}}{M} A = \tilde{A} \;\longleftrightarrow\; A = M^{-1} \tilde{A}$$

For the system of linear equations: $\quad Ax = b \leftrightsquigarrow MAx = Mb \quad$ (new system)

Example: $\quad A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \rightsquigarrow MA = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \text{---} \; \alpha_1^T \; \text{---} \\ \vdots \\ \text{---} \; \alpha_m^T \; \text{---} \end{pmatrix}$$

$$c^T = (0, \ldots, 0, c_i, 0, \ldots, 0, c_j, 0, \ldots, 0) \implies c^T A = c_i \alpha_i^T + c_j \alpha_j^T$$

Example:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} \text{---} \; \alpha_1^T \; \text{---} \\ \text{---} \; \alpha_2^T \; \text{---} \\ \text{---} \; \alpha_3^T \; \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} \; \alpha_1^T \; \text{---} \\ \text{---} \; \alpha_2^T \; \text{---} \\ \alpha_3^T + \lambda \cdot \alpha_1^T \end{pmatrix}$$

invertible with inverse: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}$

Definition: $\quad Z_{i+\lambda j} \in \mathbb{R}^{m \times m}, \; i \neq j, \; \lambda \in \mathbb{R},$

defined as the identity matrix with λ at the (i,j)th position.

Example: (exchanging rows)

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_{1 \leftrightarrow 3}} \begin{pmatrix} \text{---} \; \alpha_1^T \; \text{---} \\ \text{---} \; \alpha_2^T \; \text{---} \\ \text{---} \; \alpha_3^T \; \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} \; \alpha_3^T \; \text{---} \\ \text{---} \; \alpha_2^T \; \text{---} \\ \text{---} \; \alpha_1^T \; \text{---} \end{pmatrix}$$

Definition: $\quad P_{i \leftrightarrow j} \in \mathbb{R}^{m \times m}, \; i \neq j,$ defined as the identity matrix where the ith and the jth rows are exchanged.

Definition: (scaling rows)

$$\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} \begin{pmatrix} \text{---} \; \alpha_1^T \; \text{---} \\ \vdots \\ \text{---} \; \alpha_m^T \; \text{---} \end{pmatrix} = \begin{pmatrix} \text{---} \; d_1 \alpha_1^T \; \text{---} \\ \vdots \\ \text{---} \; d_m \alpha_m^T \; \text{---} \end{pmatrix}$$

with $d_k \neq 0$

Definition: row operations: finite combination of $\; Z_{i+\lambda j}, \; P_{i \leftrightarrow j}, \; \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix}, \ldots$

(for example: $M = Z_{3+71} \, Z_{2+81} \, P_{1 \leftrightarrow 2}$)

Property: For $A \in \mathbb{R}^{m \times n}$ and $M \in \mathbb{R}^{m \times m}$ (invertible), we have:

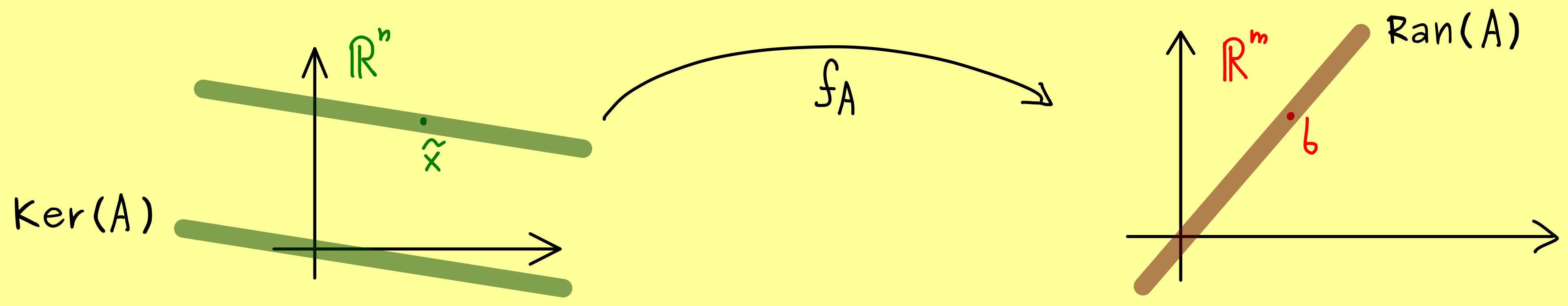
$$\operatorname{Ker}(MA) = \operatorname{Ker}(A), \quad \operatorname{Ran}(MA) = M \operatorname{Ran}(A)$$

$$:= \{ My \mid y \in \operatorname{Ran}(A) \}$$

Linear Algebra – Part 38

Set of solutions: $Ax = b$ $\quad (A \in \mathbb{R}^{m \times n})$

solution: $\tilde{x}$ satisfies $A\tilde{x} = b$



uniqueness needs $\mathrm{Ker}(A) = \{0\}$ $\qquad$ existence needs $b \in \mathrm{Ran}(A)$

Proposition: For a system $Ax = b$ $\quad (A \in \mathbb{R}^{m \times n})$

the set of solutions $S := \{ \tilde{x} \in \mathbb{R}^n \mid A\tilde{x} = b \}$

is an affine subspace (or empty).

More concretely: We have either $S = \emptyset$

or $S = v_0 + \mathrm{Ker}(A)$ for a vector $v_0 \in \mathbb{R}^n$

$= \{ v_0 + x_0 \mid x_0 \in \mathrm{Ker}(A) \}$

Proof: Assume $v_0 \in S$. $\Rightarrow A v_0 = b$

Set $\tilde{x} := v_0 + x_0$ for a vector $x_0 \in \mathbb{R}^n$.

Then: $\tilde{x} \in S \Leftrightarrow A\underbrace{\tilde{x}}_{(v_0 + x_0)} = b \Leftrightarrow \underbrace{Av_0}_{b} + Ax_0 = b$

$\Leftrightarrow Ax_0 = 0 \Leftrightarrow x_0 \in \mathrm{Ker}(A)$ $\qquad \square$

Remember: Row operations don't change the set of solutions!

$S = v_0 + \mathrm{Ker}(A)$

$= \mathrm{Ker}(MA)$

$Av_0 = b$

$\Leftrightarrow MAv_0 = Mb$

$\rightsquigarrow$ Gaussian elimination

- decide $b \in \mathrm{Ran}(A)$
- gives us a particular solution v_0
- gives us $\mathrm{Ker}(A)$

Linear Algebra – Part 39

Goal: Gaussian elimination (named after Carl Friedrich Gauß)

Solve $Ax = b$

↳ use row operations to bring $(A|b)$ into upper triangular form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

backwards substitution:

third row: $3x_3 = 1 \Rightarrow \boxed{x_3 = \frac{1}{3}}$

second row: $2x_2 + x_3 = 1 \Rightarrow \boxed{x_2 = \frac{1}{3}}$

first row: $1x_1 + 2x_2 + 3x_3 = 1 \Rightarrow x_1 = -\frac{2}{3}$

↳ or use row operations to bring $(A|b)$ into row echelon form

↳ construct solution set

Example: system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 - 1x_3 &= 4 \\ 2x_1 - 1x_2 + 7x_3 &= 0 \\ 6x_1 + 13x_2 - 4x_3 &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 2 & -1 & 7 & 0 \\ 6 & 13 & -4 & 9 \end{array}\right) \begin{array}{l} \\ -1\cdot I \\ -3\cdot I \end{array} \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 4 & -1 & -3 \end{array}\right) \begin{array}{l} \\ \\ +1\cdot II \end{array}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 7 & -7 \end{array}\right) \xrightarrow[\text{substitution}]{\text{backwards}} \begin{array}{l} x_3 = -1 \\ x_2 = -1 \\ x_1 = 3 \end{array}$$

set of solutions: $S = \left\{ \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}$

Gaussian elimination:

$$\left(\begin{array}{cccc|c} \boxed{a_{11}} & a_{12} & \cdots & a_{1n} & b_1 \\ \boxed{a_{21}} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \boxed{a_{m1}} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) = \begin{pmatrix} — & \alpha_1^T & — \\ — & \alpha_2^T & — \\ & \vdots & \\ — & \alpha_m^T & — \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \alpha_1^T \\ \alpha_2^T - \frac{a_{21}}{a_{11}} \alpha_1^T \\ \vdots \\ \alpha_m^T - \frac{a_{m1}}{a_{11}} \alpha_1^T \end{pmatrix} \xrightarrow{\text{continue iteratively}} \cdots \quad \text{row echelon form}$$

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Linear Algebra – Part 40

Row echelon form

$$A = \begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & \boxed{4} & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition: A matrix $A \in \mathbb{R}^{m \times n}$ is in row echelon form if:

(1) All zero rows (if there are any) are at the bottom.

(2) For each row: the first non-zero entry is strictly to the right of the first non-zero entry of the row above.

pivots

$$A = \begin{pmatrix} \boxed{1} & 3 & 5 & 0 \\ 0 & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \boxed{1} & 3 & 5 & 0 & 1 \\ 0 & \boxed{2} & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

variables with no pivot in their columns are called free variables (x_3)

variables with a pivot in their columns are called leading variables (x_1, x_2, x_4)

Procedure: $Ax = b \rightsquigarrow (A \mid b) \overset{\text{Gaussian elimination}}{\underset{\text{row operations}}{\rightsquigarrow}} (A' \mid b')$ row echelon form

$\rightsquigarrow$ put free variable to the right-hand side $\rightsquigarrow$ backwards substitution $\rightsquigarrow$ solutions S

Example:

free variables

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\rightsquigarrow \begin{array}{ccc|cl} x_1 & x_3 & x_4 & & \\ \boxed{1} & 0 & 1 & 3 - 2x_2 & \mathrm{I} \\ 0 & \boxed{2} & -1 & 2 - 4x_5 & \mathrm{II} \\ 0 & 0 & \boxed{4} & 8 - 8x_5 & \mathrm{III} \\ 0 & 0 & 0 & 0 & \end{array}$$

III $4x_4 = 8 - 8x_5 \Rightarrow \boxed{x_4 = 2 - 2x_5}$ $x_5 \in \mathbb{R}$

II $2x_3 - x_4 = 2 - 4x_5$

$\Rightarrow 2x_3 - 2 + 2x_5 = 2 - 4x_5 \Rightarrow 2x_3 = 4 - 6x_5 \Rightarrow \boxed{x_3 = 2 - 3x_5}$

I $x_1 + x_4 = 3 - 2x_2 \Rightarrow x_1 + 2 - 2x_5 = 3 - 2x_2 \Rightarrow \boxed{x_1 = 1 - 2x_2 + 2x_5}$

set of solutions: $$S = \left\{ \begin{pmatrix} 1 - 2x_2 + 2x_5 \\ x_2 \\ 2 - 3x_5 \\ 2 - 2x_5 \\ x_5 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$

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Linear Algebra – Part 41

$$A \in \mathbb{R}^{m \times n} \quad \overset{\text{Gaussian elimination}}{\rightsquigarrow} \quad \text{row echelon form}$$

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{2} & -1 & 4 & 0 \\ 0 & 0 & 0 & \boxed{4} & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad \mathrm{Ker}(A) = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}$$

Remember:

$\dim(\mathrm{Ker}(A)) =$ number of free variables
$+$
$\dim(\mathrm{Ran}(A)) =$ number of leading variables
$= n$

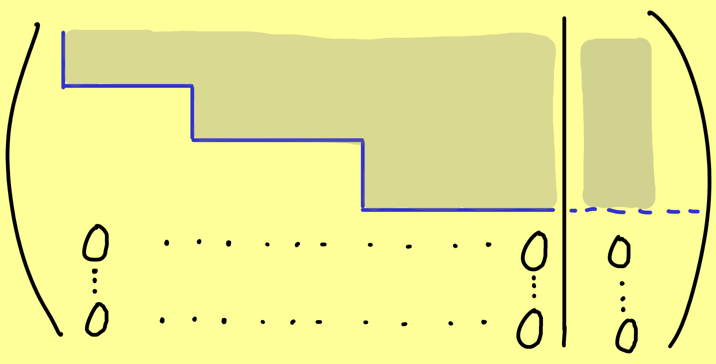
Proposition: For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, we have the following equivalences:

(1) $Ax = b$ has at least one solution.

(2) $b \in \mathrm{Ran}(A)$

(3) b can be written as a linear combination of the columns of A.

(4) Row echelon form looks like:



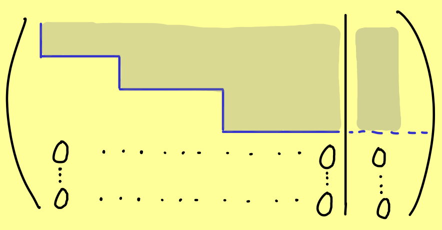
Proof: (1) $\Leftrightarrow$ (2) given by definition of $\mathrm{Ran}(A)$

(2) $\Leftrightarrow$ (3) given by column picture of $\mathrm{Ran}(A)$

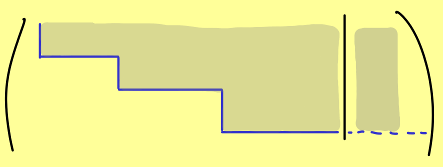
$$\mathrm{Ran}(A) = \left\{ \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix} x \;\middle|\; x \in \mathbb{R}^n \right\}$$
$$= \left\{ x_1 \cdot \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \;\middle|\; x \in \mathbb{R}^n \right\}$$

(4) $\Rightarrow$ (1)

Assume we have this:



Then solve

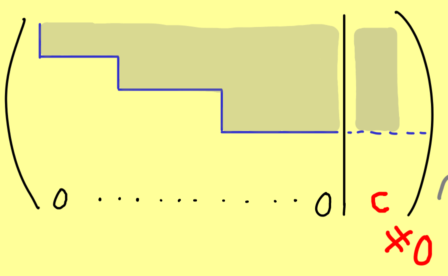


by backwards substitution.

(or argue with $\mathrm{rank}(A) = \mathrm{rank}((A|b))$)

(1) $\Rightarrow$ (4) (let's show: $\neg(4) \Rightarrow \neg(1)$)

Assume:

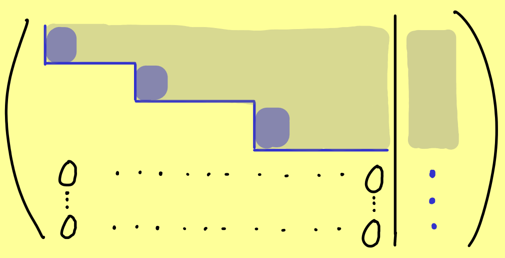


$c \neq 0$ not solvable $\quad 0 = c \;$ ↯

$\Rightarrow$ no solution for $Ax = b$ $\quad \square$

Linear Algebra – Part 42

$Ax = b \rightsquigarrow$ row echelon form



$S = \emptyset$ or $S = v_0 + \operatorname{Ker}(A)$

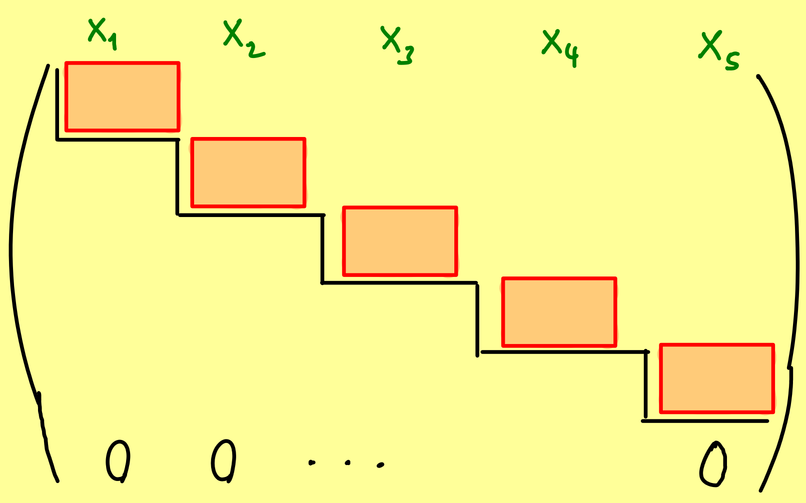
<u>Proposition:</u> For $A \in \mathbb{R}^{m \times n}$, we have the following equivalences:

(a) For every $b \in \mathbb{R}^m$: $Ax = b$ has at most one solution.

(b) $\operatorname{Ker}(A) = \{0\}$

(c) Row echelon form looks like:

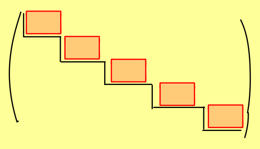
$\boxed{\text{every column has a pivot}}$



(d) $\operatorname{rank}(A) = n$

(e) The linear map $f_A : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ is injective.

<u>Result for square matrices:</u> For $A \in \mathbb{R}^{n \times n}$:



$\operatorname{Ker}(A) = \{0\} \iff \operatorname{Ran}(A) = \mathbb{R}^n \iff Ax = b$ has a <u>unique solution</u> for some $b \in \mathbb{R}^n$

$\Updownarrow$ $\qquad$ $\Updownarrow$

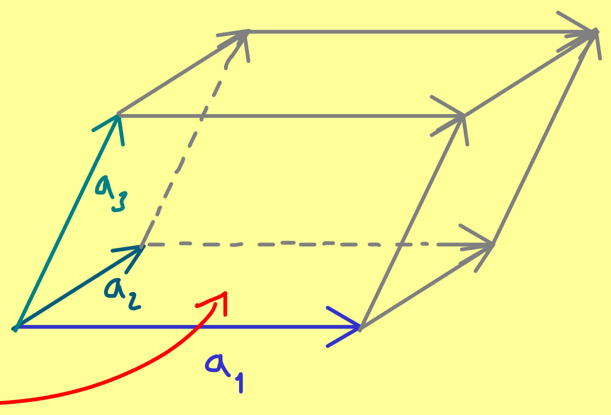
f_A injective $\iff$ f_A surjective $\iff$ $Ax = b$ has a <u>unique solution</u> for all $b \in \mathbb{R}^n$

Linear Algebra – Part 43

$A \in \mathbb{R}^{n \times n} \rightsquigarrow \det(A) \in \mathbb{R}$ with properties:

(1) $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$, columns span a parallelepiped

volume $= |\det(A)|$



(2) $\det(A) = 0 \iff \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix}, \dots, \begin{pmatrix} | \\ a_n \\ | \end{pmatrix}$ linearly dependent

$\iff A$ is not invertible

(3) sign of $\det(A)$ gives orientation $\left(\det(\mathbb{1}_n) = +1 \right)$

Linear Algebra – Part 44

$A \in \mathbb{R}^{2\times 2} \rightsquigarrow$ system of linear equations $Ax = b$

Assume $a_{11} \neq 0$

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right) \overset{\mathrm{II} - \frac{a_{21}}{a_{11}} \mathrm{I}}{\rightsquigarrow} \left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array}\right) \overset{\mathrm{II} \cdot a_{11}}{\rightsquigarrow}$$

$$\rightsquigarrow \left(\begin{array}{cc|c} \boxed{a_{11}} & a_{12} & b_1 \\ 0 & \boxed{a_{11}a_{22} - a_{21}a_{12}} & a_{11} \cdot b_2 - a_{21} b_1 \end{array}\right)$$

$\neq 0 \iff$ we have a unique solution

Definition: For a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, the number

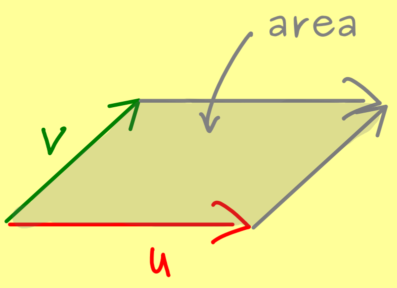
$$\det(A) := a_{11} a_{22} - a_{12} a_{21}$$

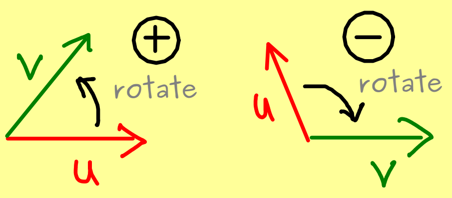
is called the determinant of A.

What about volumes? $\rightsquigarrow$ vol_n

in $\mathbb{R}^2$: $\mathrm{vol}_2(u, v) :=$ orientated area of parallelogram

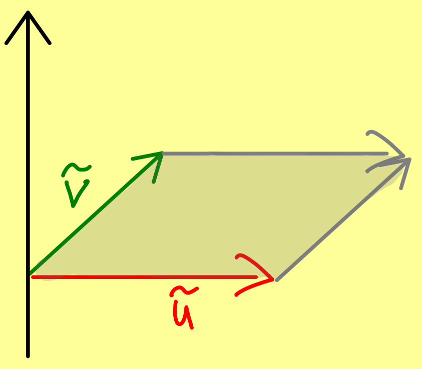
$\pm$





Relation to cross product: embed $\mathbb{R}^2$ into $\mathbb{R}^3$: $\tilde{u} := \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$, $\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

$\mathbb{R}^3$:



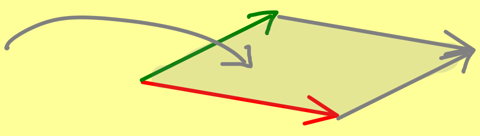
$$\| \tilde{u} \times \tilde{v} \| = \left\| \begin{pmatrix} 0 \\ 0 \\ u_1 v_2 - v_1 u_2 \end{pmatrix} \right\| = \underbrace{| u_1 v_2 - v_1 u_2 |}_{\det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}}$$

Result: $\mathrm{vol}_2(u, v) = \det\begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$ (volume function = determinant)

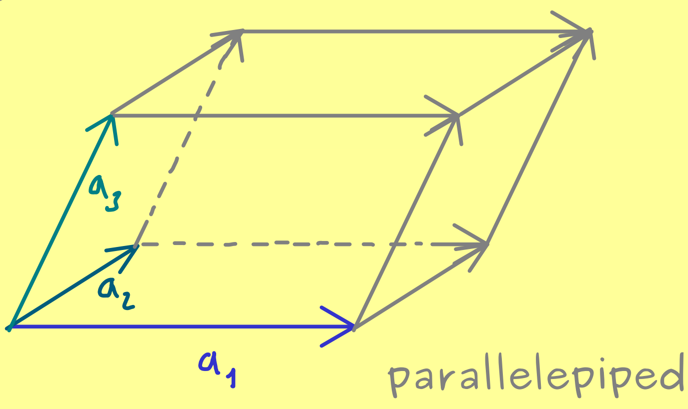
Linear Algebra – Part 45

volume measure?

- area in $\mathbb{R}^2$

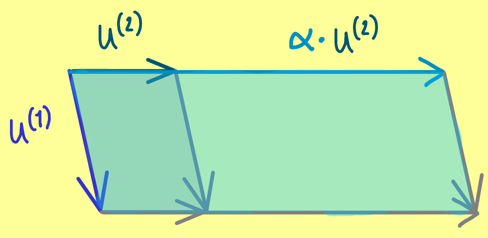


- n-dimensional volume $\mathbb{R}^n$



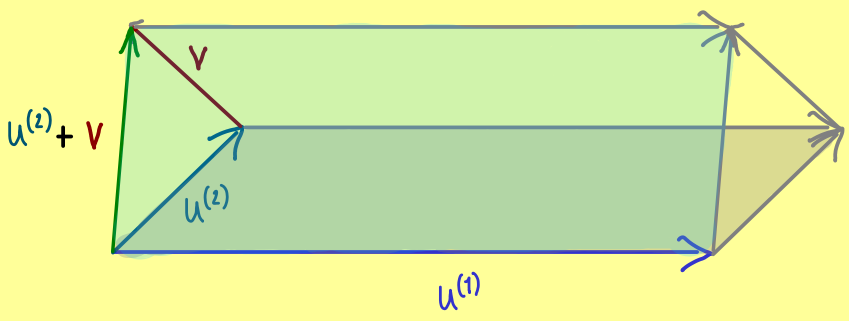
Definition: $\mathrm{vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$ is called n-dimensional volume function if

(a) $$\mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, \alpha \cdot u^{(j)}, \dots, u^{(n)}\right) = \alpha \cdot \mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(n)}\right)$$



for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$
for all $\alpha \in \mathbb{R}$
for all $j \in \{1, \dots, n\}$

(b) $$\mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, u^{(j)} + v, \dots, u^{(n)}\right) = \mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(n)}\right) + \mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, v, \dots, u^{(n)}\right)$$



for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$
for all $v \in \mathbb{R}^n$
for all $j \in \{1, \dots, n\}$

(c) $$\mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, u^{(i)}, \dots, u^{(j)}, \dots, u^{(n)}\right)$$
$$= - \mathrm{vol}_n\left(u^{(1)}, u^{(2)}, \dots, u^{(j)}, \dots, u^{(i)}, \dots, u^{(n)}\right)$$

for all $u^{(1)}, \dots, u^{(n)} \in \mathbb{R}^n$
for all $i, j \in \{1, \dots, n\}$
$i \neq j$

(d) $$\mathrm{vol}_n\left(e_1, e_2, \dots, e_n\right) = 1$$ (unit hypercube)

Result in $\mathbb{R}^2$:

$$\mathrm{vol}_2\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) = \mathrm{vol}_2\left(\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$$

$$\overset{(b)}{=} \mathrm{vol}_2\left(\begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) + \mathrm{vol}_2\left(\begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$$

$$\overset{(a)}{=} a \cdot \mathrm{vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) + c \cdot \mathrm{vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$$

$$\overset{(b)}{=} a \cdot \mathrm{vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}\right) + a \cdot \mathrm{vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix}\right) + c \cdot \mathrm{vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix}\right) + c \cdot \mathrm{vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ d \end{pmatrix}\right)$$

$$\overset{(b)}{=} a \cdot b \underbrace{\mathrm{vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{=0} + a \cdot d \underbrace{\mathrm{vol}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{\overset{(d)}{=} 1} + c \cdot b \underbrace{\mathrm{vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{= -1} + c \cdot d \cdot \underbrace{\mathrm{vol}_2\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{=0}$$

$$\overset{(c),(d)}{=} a \cdot d - b \cdot c = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Define: $$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \mathrm{vol}_n\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}\right)$$

Linear Algebra – Part 46

n-dimensional volume form: $\mathrm{vol}_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$

- linear in each entry
- antisymmetric
- $\mathrm{vol}_n(e_1, e_2, \dots, e_n) = 1$

Let's calculate:

$$\mathrm{vol}_n\left(\begin{pmatrix}a_{11}\\ \vdots\\ a_{n1}\end{pmatrix}, \underbrace{\begin{pmatrix}a_{12}\\ \vdots \\ a_{n2}\end{pmatrix}, \dots, \begin{pmatrix}a_{1n}\\ \vdots\\ a_{nn}\end{pmatrix}}_{(*)}\right) = \mathrm{vol}_n\Big(a_{11}\cdot e_1 + \cdots + a_{n1}\, e_n, (*)\Big)$$

$$= a_{11}\cdot \mathrm{vol}_n\big(e_1, (*)\big) + \cdots + a_{n1}\cdot \mathrm{vol}_n\big(e_n, (*)\big)$$

$$= \sum_{j_1=1}^{n} a_{j_1,1}\, \mathrm{vol}_n\big(e_{j_1}, (*)\big) = \sum_{j_1=1}^{n} a_{j_1,1}\, \mathrm{vol}_n\left(e_{j_1}, \begin{pmatrix}a_{12}\\ \vdots \\ a_{n2}\end{pmatrix}, \dots, \begin{pmatrix}a_{1n}\\ \vdots\\ a_{nn}\end{pmatrix}\right)$$

$$= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} a_{j_1,1}\, a_{j_2,2}\cdot \mathrm{vol}_n\left(e_{j_1}, e_{j_2}, \begin{pmatrix}a_{13}\\ \vdots \\ a_{n3}\end{pmatrix}, \dots, \begin{pmatrix}a_{1n}\\ \vdots\\ a_{nn}\end{pmatrix}\right)$$

$$= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_n=1}^{n} a_{j_1,1}\, a_{j_2,2}\cdots a_{j_n,n}\cdot \underbrace{\mathrm{vol}_n\big(e_{j_1}, e_{j_2}, \dots, e_{j_n}\big)}_{=0 \text{ if two indices coincide}}$$

permutation of $\{1,\dots,n\}$

$$= \sum_{(j_1,\dots,j_n)\in S_n} a_{j_1,1}\, a_{j_2,2}\cdots a_{j_n,n}\cdot \underbrace{\mathrm{vol}_n\big(e_{j_1}, e_{j_2}, \dots, e_{j_n}\big)}_{=\left\{\begin{smallmatrix}1\\ -1\end{smallmatrix}\right\}}$$

where all entries are different

set of all permutations of $\{1,\dots,n\}$

$$\mathrm{sgn}\big((j_1,\dots,j_n)\big) = \begin{cases} +1, & \text{even number of exchanges to get to } (1,\dots,n) \\ -1, & \text{odd number of exchanges to get to } (1,\dots,n) \end{cases}$$

$$= \sum_{(j_1,\dots,j_n)\in S_n} \mathrm{sgn}\big((j_1,\dots,j_n)\big)\, a_{j_1,1}\, a_{j_2,2}\cdots a_{j_n,n} = \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

(Leibniz formula)

Linear Algebra – Part 47

Leibniz formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix} = \sum_{(j_1,\dots,j_n)\in S_n} \operatorname{sgn}((j_1,\dots,j_n))\, a_{j_1,1}\, a_{j_2,2} \cdots a_{j_n,n}$$

how many terms?

For $n=2$: $(1,2)$, $(2,1)$ 2 permutations $\left(\times\right)$

For $n=3$: $(1,2,3),(2,3,1),(3,1,2)$ $(1,3,2),(3,2,1),(2,1,3)$ 6 permutations (rule of Sarrus)

For $n=4$: ... 24 permutations

For n: ... $n!$ permutations

Rule of Sarrus:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = + \quad + \quad + \quad - \quad - \quad -$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

Example:

$$\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & 4 & 1\end{pmatrix} = \underline{-1} + 8 + \underline{(-4)}\ \underline{-(-1)}\ \underline{-(-8)}\ \underline{-4} = 8$$

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Linear Algebra – Part 48

4x4-matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{11}\cdot\det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$

6 permutations

24 permutations

$$- a_{21}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} & \cancel{a_{24}} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

6 permutations

checkerboard

$$\begin{pmatrix} + & - & + & \\ - & + & - & \\ + & - & & \ddots \\ - & + & & \end{pmatrix}$$

$$+ a_{31}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} & \cancel{a_{34}} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

6 permutations

$$- a_{41}\cdot\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \cancel{a_{41}} & \cancel{a_{42}} & \cancel{a_{43}} & \cancel{a_{44}} \end{pmatrix}$$

6 permutations

Idea: $n\times n \rightsquigarrow (n-1)\times(n-1) \rightsquigarrow \cdots \rightsquigarrow 3\times 3 \rightsquigarrow 2\times 2 \rightsquigarrow 1\times 1$

Laplace expansion: $A \in \mathbb{R}^{n\times n}$. For jth column:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij}\cdot\det\left(A^{(i,j)}\right)$$ expanding along the jth column

For ith row:

ith row and jth column are deleted

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\cdot\det\left(A^{(i,j)}\right)$$ expanding along the ith row

Example:

expaning along 2nd row

$$\det\begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 6 & 0 & 1 & 2 \end{pmatrix} = -2\cdot\det\begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= (-2)(-1)\cdot 1\cdot\det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2\cdot(6-4) = 4$$

Linear Algebra – Part 49

Triangular matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ & a_{22} & \ddots & & \vdots \\ & & a_{33} & & \vdots \\ & 0 & & \ddots & \vdots \\ & & & & a_{nn} \end{pmatrix} = a_{11} \cdot a_{22} \cdots a_{nn}$$

Block matrices:

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} & b_{11} & b_{12} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ a_{m1} & \cdots & a_{mm} & b_{m1} & \cdots & & b_{mk} \\ 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots & c_{21} & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & c_{k1} & & \cdots & c_{kk} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

$$\Rightarrow \quad \det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \cdot \det(C)$$

Proposition: $\det(A^T) = \det(A)$

Proposition: $A, B \in \mathbb{R}^{n \times n}$: $\boxed{\det(A \cdot B) = \det(A) \cdot \det(B)}$ multiplicative map

If A is invertible, then: $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$$\det(A^{-1} B A) = \det(B)$$

Linear Algebra – Part 50

determinant is multiplicative: $\det(MA) = \det(M)\cdot\det(A)$

Gaussian elimination: $A \xrightarrow[\text{row operations}]{} MA$ (see part 37)

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}}_{Z_{3+\lambda 1}} \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ — \alpha_3^T — \end{pmatrix} = \begin{pmatrix} — \alpha_1^T — \\ — \alpha_2^T — \\ \alpha_3^T + \lambda\cdot\alpha_1^T \end{pmatrix}$$

$$Z_{3+\lambda 1} \Rightarrow \det(Z_{3+\lambda 1}) = 1$$

Adding rows with $Z_{i+\lambda j}$ $(i \neq j,\ \lambda \in \mathbb{R})$ does not change the determinant!

Exchanging rows with $P_{i \leftrightarrow j}$ $(i \neq j)$ does change the sign of the determinant!

Scaling one row with factor d_j scales the determinant by d_j!

Column operations? $\det(A^T) = \det(A)$ ✓

Example:

$$\det\begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 & 4 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix} \overset{\substack{\text{rows} \\ II - 1\cdot V}}{=} \det\begin{pmatrix} -1 & 1 & 0 & -2 & 0 \\ 0 & 4 & 0 & -2 & 2 \\ 1 & 0 & 0 & -3 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & -2 & 1 & 1 & 2 \end{pmatrix}$$

$$\overset{\text{Laplace expansion}}{=} (+1)\cdot\det\begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 4 & -2 & 2 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

$$\overset{\substack{\text{columns} \\ II - 2\,IV \\ III + IV}}{=} \det\begin{pmatrix} -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 1 & -4 & 3 & 3 \end{pmatrix}$$

$$\overset{\text{Laplace expansion}}{=} (+2)\cdot\det\begin{pmatrix} -1 & 1 & -2 \\ 1 & -2 & -2 \\ 1 & -4 & 3 \end{pmatrix} = 2\cdot 13 = 26$$

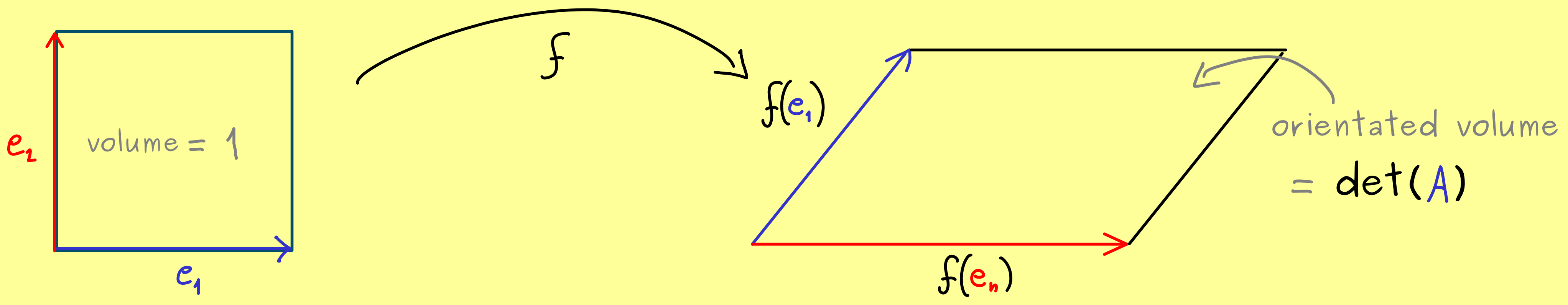
Linear Algebra – Part 51

matrix $A \in \mathbb{R}^{n\times n}$ $\rightsquigarrow$ linear map $f_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$

linear map $f: \mathbb{R}^n \to \mathbb{R}^n$ $\rightsquigarrow$ there is exactly one $A \in \mathbb{R}^{n\times n}$ with $f = f_A$

Here: $A = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}$

unit cube in $\mathbb{R}^n$



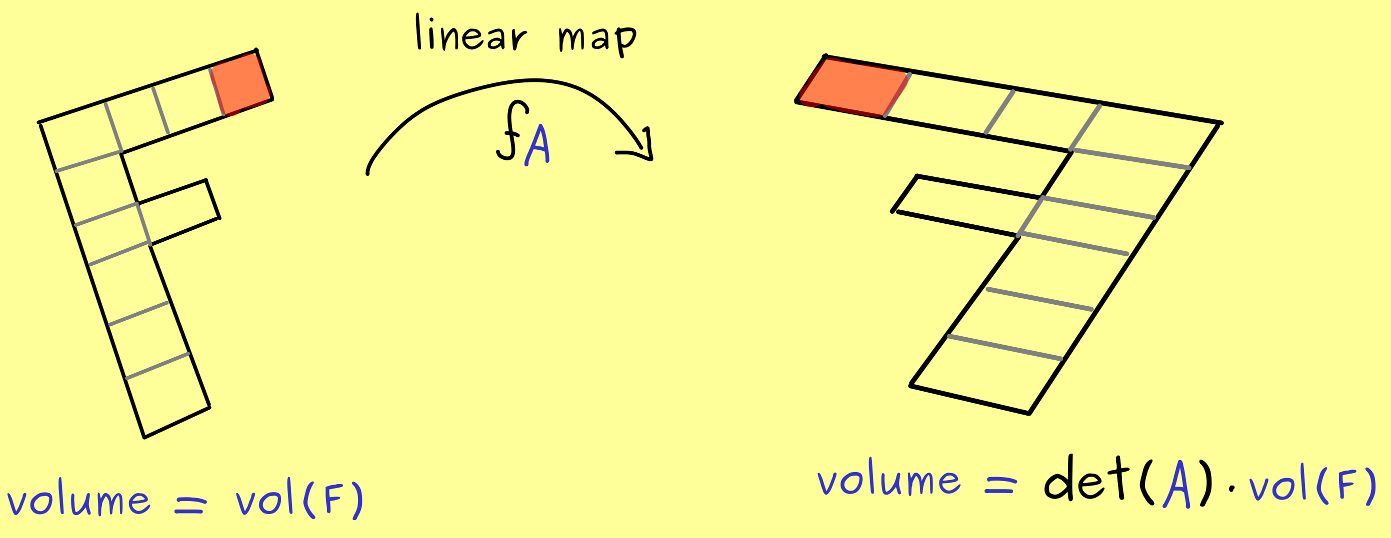
Remember: $\det(A)$ gives the relative change of volume caused by f_A.

Definition: For a linear map $f: \mathbb{R}^n \to \mathbb{R}^n$, we define the determinant:

$$\det(f) := \det(A) \quad \text{where } A \text{ is } \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}$$

Multiplication rule: $\det(f \circ g) = \det(f)\det(g)$

Volume change:



Linear Algebra – Part 52

We know for $A \in \mathbb{R}^{2\times 2}$: $\det(A) \neq 0 \iff Ax = b$ has a unique solution

$\iff A$ invertible $=$ non-singular

For $A \in \mathbb{R}^{n\times n}$: $\det(A) = 0 \iff A$ singular

Proposition: For $A \in \mathbb{R}^{n\times n}$, the following claims are equivalent:

- $\det(A) \neq 0$
- columns of A are linearly independent
- rows of A are linearly independent
- $\operatorname{rank}(A) = n$
- $\operatorname{Ker}(A) = \{0\}$
- A is invertible
- $Ax = b$ has a unique solution for each $b \in \mathbb{R}^n$

Cramer's rule: $A \in \mathbb{R}^{n\times n}$ non-singular, $b \in \mathbb{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ unique solution of $Ax = b$.

Then:

$$x_i = \frac{\det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & b & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}}{\det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & a_i & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}}$$

Proof: Use cofactor matrix $C \in \mathbb{R}^{n\times n}$ defined: $c_{ij} = (-1)^{i+j} \cdot \det(A)$ jth column deleted, ith row deleted

$$\overset{\text{Laplace expansion}}{=} \det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{j-1} & e_i & a_{j+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix}$$

We can show: $A^{-1} = \dfrac{C^T}{\det(A)}$

Hence: $x = A^{-1} b = \dfrac{C^T b}{\det(A)}$ and $(C^T b)_i = \sum_{k=1}^{n} (C^T)_{ik} b_k = \sum_{k=1}^{n} c_{ki} b_k$

$$= \sum_{k=1}^{n} \det\begin{pmatrix} | & & | & | & | & & | \\ a_1 & \cdots & a_{i-1} & e_k & a_{i+1} & \cdots & a_n \\ | & & | & | & | & & | \end{pmatrix} b_k$$

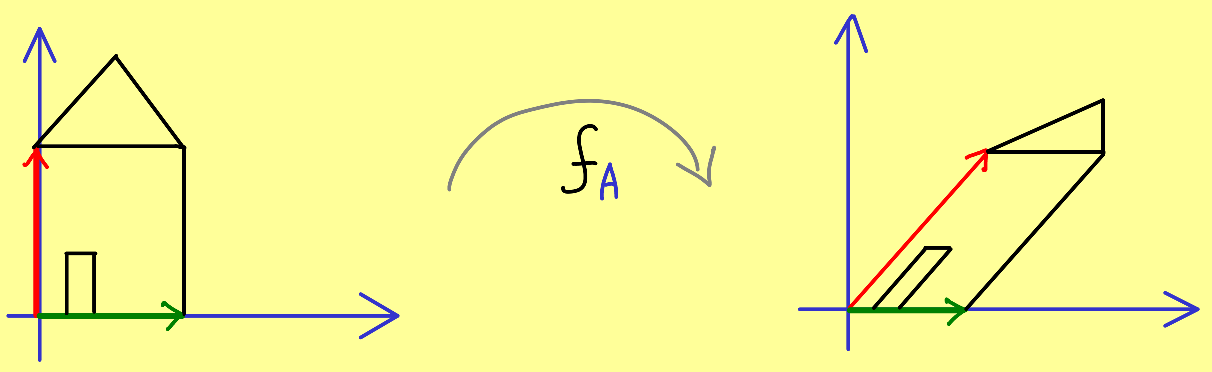
$$\overset{\text{linear in the } i\text{th column}}{=} \det\begin{pmatrix} | & & | & b_1 & | & & | \\ a_1 & \cdots & a_{i-1} & b_2 & a_{i+1} & \cdots & a_n \\ | & & | & \vdots & | & & | \\ & & & b_n & & & \end{pmatrix}$$

$\square$

Linear Algebra – Part 53

eigenvalue (German: Eigenwert) (David Hilbert, 1904)
↳ proper/own/characteristic

Consider: $A \in \mathbb{R}^{n\times n} \rightsquigarrow f_A : \mathbb{R}^n \to \mathbb{R}^n$ linear map



Question: Are there vectors which are only scaled by f_A?

Answer: $Ax = \lambda \cdot x$ for a number $\lambda \in \mathbb{R}$

$\Leftrightarrow (A - \lambda \mathbb{1})x = 0$ for a number $\lambda \in \mathbb{R}$

$\Leftrightarrow x \in \operatorname{Ker}(A - \lambda \mathbb{1})$ for a number $\lambda \in \mathbb{R}$

eigenvector (if $x \neq 0$) eigenvalue

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow \begin{matrix} x_1 + x_2 = \lambda \cdot x_1 & \text{I} \\ x_2 = \lambda \cdot x_2 & \text{II} \end{matrix}$

For II: $\boxed{\lambda = 1}$ or $\underbrace{x_2 = 0}_{\text{I}} \Rightarrow x_1 = \lambda \cdot x_1 \Rightarrow \boxed{\lambda = 1}$ or $x_1 = 0$

For I: $x_1 + x_2 = x_1 \Rightarrow x_2 = 0$

Solution: eigenvalue: $\lambda = 1$

eigenvectors: $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ for $x_1 \in \mathbb{R} \setminus \{0\}$

Definition: $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$.

If there is $x \in \mathbb{R}^n \setminus \{0\}$ with $Ax = \lambda x$, then:

- λ is called an eigenvalue of A
- x is called an eigenvector of A (associated to λ)
- $\operatorname{Ker}(A - \lambda \mathbb{1})$ eigenspace of A (associated to λ)

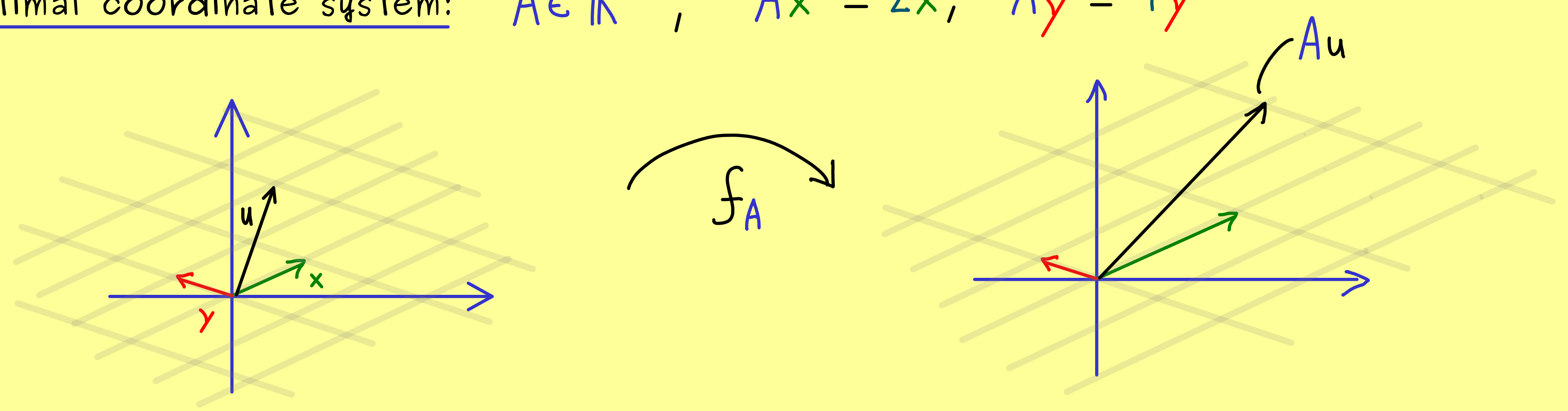
The set of all eigenvalues of A : $\operatorname{spec}(A)$ spectrum of A

Linear Algebra – Part 54

$A \in \mathbb{R}^{n \times n} \leftrightsquigarrow f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear map

eigenvalue equation: $Ax = \lambda \cdot x \quad , \quad x \neq 0$

optimal coordinate system: $A \in \mathbb{R}^{2\times 2}, \quad Ax = 2x, \quad Ay = 1y$



$u = a \cdot x + b \cdot y$

$$\begin{aligned} Au &= A(a\cdot x + b \cdot y) \\ &= a \cdot Ax + b\, Ay \\ &= 2ax + 1by \end{aligned}$$

How to find enough eigenvectors?

$x \neq 0$ eigenvector associated to eigenvalue $\lambda \iff x \in \operatorname{Ker}(\underbrace{A - \lambda \mathbb{1}}_{\text{singular matrix}})$

$$\det(A - \lambda \mathbb{1}) = 0 \iff \operatorname{Ker}(A - \lambda\mathbb{1}) \text{ is non-trivial}$$

$$\iff \lambda \text{ is eigenvalue of } A$$

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad A - \lambda \mathbb{1} = \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix}$

$$\begin{aligned} \det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} &= (3-\lambda)(4-\lambda) - 2 \\ &= 10 - 7\lambda + \lambda^2 \\ &= (\lambda - 5)(\lambda - 2) \overset{!}{=} 0 \end{aligned}$$

characteristic polynomial

$\Rightarrow$ 2 and 5 are eigenvalues of A

General case: For $A \in \mathbb{R}^{n\times n}$:

$$\det(A - \lambda\mathbb{1}) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} - \lambda \end{pmatrix}$$

Leibniz formula

$$\overset{\text{Leibniz formula}}{=} (a_{11} - \lambda) \cdots (a_{nn} - \lambda) + \cdots$$

$$= (-1)^n \cdot \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda^1 + c_0$$

Definition: For $A \in \mathbb{R}^{n\times n}$, the polynomial of degree n given by

$$p_A : \quad \lambda \longmapsto \det(A - \lambda \mathbb{1})$$

is called the characteristic polynomial of A.

Remember: The zeros of the characteristic polynomial are exactly the eigenvalues of A.

Linear Algebra – Part 55

$$\lambda \in \operatorname{spec}(A) \iff \det(A - \lambda \mathbb{1}) = 0$$

Fundamental theorem of algebra: For $a_n \neq 0$ and $a_n, a_{n-1}, \dots, a_0 \in \mathbb{C}$, we have:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

has n solutions $x_1, x_2, \dots, x_n \in \mathbb{C}$ (not necessarily distinct).

Hence: $p(x) = a_n (x - x_n) \cdot (x - x_{n-1}) \cdots (x - x_1)$

Conclusion for characteristic polynomial: $A \in \mathbb{R}^{n \times n}$, $p_A(\lambda) := \det(A - \lambda \mathbb{1})$

- $p_A(\lambda) = 0$ has at least one solution in $\mathbb{C}$

  $\Rightarrow$ A has at least one eigenvalue in $\mathbb{C}$

  Example: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow p_A(\lambda) = \lambda^2 + 1$

  $\Rightarrow -i$ and i are eigenvalues

- $p_A(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

  Example: $A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 2 \end{pmatrix} \Rightarrow p_A(\lambda) = (\lambda - 1)^2 (\lambda - 2)^2$

Definition: If $\tilde{\lambda}$ occurs k times in the factorisation $p_A(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$,

then we say: $\tilde{\lambda}$ has algebraic multiplicity $k =: \alpha(\tilde{\lambda})$

Remember:
- If $\tilde{\lambda} \in \operatorname{spec}(A) \iff 1 \leq \alpha(\tilde{\lambda}) \leq n$
- $\displaystyle\sum_{\tilde{\lambda} \in \mathbb{C}} \alpha(\tilde{\lambda}) = n$

The Bright Side of Mathematics

Linear Algebra – Part 56

eigenvalues: $\lambda \in \operatorname{spec}(A) \iff \underbrace{\det(A-\lambda \mathbb{1})}_{\text{characteristic polynomial}} = 0$

Next step for a given $\lambda \in \operatorname{spec}(A)$:

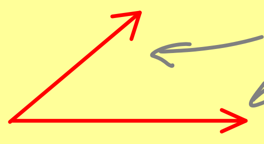
$$\operatorname{Ker}(A-\lambda\mathbb{1}) \supsetneq \{0\}$$

Solve: $\left(\begin{array}{cccc|c} a_{11}-\lambda & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22}-\lambda & & \vdots & 0 \\ \vdots & & \ddots & & \vdots \\ a_{n1} & \cdots & & a_{nn}-\lambda & 0 \end{array}\right)$

Solution set: eigenspace (associated to λ)

Definition: $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$ eigenvalue

$$\gamma(\lambda) := \dim(\operatorname{Ker}(A-\lambda\mathbb{1}))$$ geometric multiplicity of λ



eigenvectors span eigenspace

Example: $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ characteristic polynomial:

$$\det(A-\lambda\mathbb{1}) = (2-\lambda)(2-\lambda)(3-\lambda) = (2-\lambda)^2(3-\lambda)$$

$$\Longrightarrow \operatorname{spec}(A) = \{2, 3\}$$

algebraic multiplicity 2   algebraic multiplicity 1

$$\operatorname{Ker}(A-2\cdot\mathbb{1}) = \operatorname{Ker}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

solve system: $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \overset{\text{exchange II and III}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \rightsquigarrow x_2 = 0 \\ \rightsquigarrow x_3 = 0 \end{array}$

x_1 free variable

backwards substitution

solution set: $\left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$

eigenvector

$$\Longrightarrow \text{geometric multiplicity } \gamma(2) = 1 < \alpha(2)$$

Linear Algebra – Part 57

Recall:
$\det(A - \lambda \mathbb{1}) = 0$
$\Leftrightarrow$
$\lambda \in \operatorname{spec}(A)$

Proposition:

(a) $$\operatorname{spec}\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & & & a_{2n} \\ & & \ddots & & \vdots \\ & & & & a_{nn} \end{pmatrix} = \{a_{11}, a_{22}, \dots, a_{nn}\}$$

(b) $$\operatorname{spec}\begin{pmatrix} \boxed{B} & C \\ 0 & \boxed{D} \end{pmatrix} = \operatorname{spec}(B) \cup \operatorname{spec}(D)$$ (part 49)

B: m×m matrix, D: k×k matrix

(c) $$\operatorname{spec}(A^T) = \operatorname{spec}(A)$$

Example:

(a) $$\operatorname{spec}\begin{pmatrix} 2 & 5 & 8 & 9 \\ 0 & 3 & 0 & 8 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \{1, 2, 3\}$$

algebraic multiplicity is 2

(b) $$\operatorname{spec}\begin{pmatrix} 1 & 2 & 4 & 5 & 8 & 7 \\ 0 & 7 & 7 & 9 & 8 & 4 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 \\ 0 & 0 & 5 & 6 & 1 & 2 \\ 0 & 0 & 7 & 9 & 0 & 3 \end{pmatrix} = \operatorname{spec}\begin{pmatrix} 1 & 2 \\ 0 & 7 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 5 & 0 & 0 & 0 \\ 7 & 8 & 0 & 0 \\ 5 & 6 & 1 & 2 \\ 7 & 9 & 0 & 3 \end{pmatrix}$$

$$= \{1, 7\} \cup \operatorname{spec}\begin{pmatrix} 5 & 0 \\ 7 & 8 \end{pmatrix} \cup \operatorname{spec}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$= \{1, 7, 5, 8, 1, 3\}$$

$$= \{1, 3, 5, 7, 8\}$$

algebraic multiplicity is 2

Linear Algebra – Part 58

$\mathrm{spec}(A) \subseteq \mathbb{C}$ (fundamental theorem of algebra)

$\hookrightarrow$ Consider $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$

Definition: $\mathbb{C}^n$: column vectors with n entries from $\mathbb{C}$ $\left(\begin{pmatrix} i+2 \\ 1 \end{pmatrix} \in \mathbb{C}^2 \right)$

$\mathbb{C}^{m \times n}$: matrices with $m \times n$ entries from $\mathbb{C}$ $\left(\begin{pmatrix} i & i-1 \\ 0 & 2 \end{pmatrix} \in \mathbb{C}^{2 \times 2} \right)$

Operations like before: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ + in $\mathbb{C}$

$\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$ $\cdot$ in $\mathbb{C}$

Properties: The set $\mathbb{C}^n$ together with $+, \cdot$ is a complex vector space:

(a) $(\mathbb{C}^n, +)$ is an abelian group:

(1) $u + (v + w) = (u + v) + w$ (associativity of $+$)

(2) $v + 0 = v$ with $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ (neutral element)

(3) $v + (-v) = 0$ with $-v = \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix}$ (inverse elements)

(4) $v + w = w + v$ (commutativity of $+$)

(b) scalar multiplication is compatible: $\cdot : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$

(5) $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$

(6) $1 \cdot v = v$

(c) distributive laws:

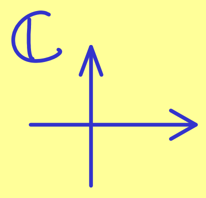
(7) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$

(8) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$

$\rightsquigarrow$ same notions: subspace, span, linear independence, basis, dimension, ...

Remember: $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \dots, \; e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ basis of $\mathbb{C}^n$

$\Longrightarrow \dim(\mathbb{C}^n) = n$ $\quad (\dim(\mathbb{C}^1) = 1)$



complex dimension

Standard inner product: $u, v \in \mathbb{C}^n$: $\langle u, v \rangle = \overline{u}_1 \cdot v_1 + \overline{u}_2 \cdot v_2 + \dots + \overline{u}_n \cdot v_n$

standard norm $\longrightarrow$ $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{|u_1|^2 + \dots + |u_n|^2}$

Example: $\left\| \begin{pmatrix} i \\ -1 \end{pmatrix} \right\| = \sqrt{|i|^2 + |-1|^2} = \sqrt{2}$

Linear Algebra – Part 59

Recall: in $\mathbb{R}^n$: $\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$

in $\mathbb{C}^n$: $\langle x,y\rangle = \sum_{k=1}^{n} \overline{x_k}\, y_k$

in $\mathbb{R}^n$: $\langle x,Ay\rangle = \langle A^T x,y\rangle$

$$\langle x,Ay\rangle = \sum_{k=1}^{n} x_k (Ay)_k = \sum_{\substack{k=1\\ j=1}}^{n} x_k a_{kj} y_j = \sum_{\substack{k=1\\ j=1}}^{n} (A^T)_{jk} x_k y_j = \langle A^T x, y\rangle$$

in $\mathbb{C}^n$:
$$\langle x,Ay\rangle = \sum_{\substack{k=1\\ j=1}}^{n} \overline{x_k}\, a_{kj} y_j = \sum_{\substack{k=1\\ j=1}}^{n} a_{kj}\overline{x_k}\, y_j = \sum_{\substack{k=1\\ j=1}}^{n} \overline{\left(\overline{A^T}\right)_{jk} x_k}\; y_j$$

$$= \langle A^* x, y\rangle$$

Definition: For $A \in \mathbb{C}^{m\times n}$ with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & & \ddots & & \vdots \\ \vdots & & & & a_{mn} \end{pmatrix}$,

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \ddots & & \vdots \\ \vdots & & & \\ \overline{a_{1n}} & \cdots & & \overline{a_{mn}} \end{pmatrix} \in \mathbb{C}^{n\times m}$$

is called the adjoint matrix/ conjugate transpose/ Hermitian conjugate.

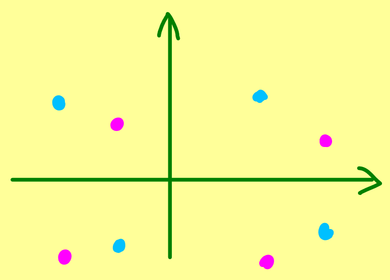
Examples: (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} i & 1+i & 0 \\ 2 & e^{-i} & 1-i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} -i & 2 \\ 1-i & e^{i} \\ 0 & 1+i \end{pmatrix}$

Remember: in $\mathbb{R}^n$: $\langle x,y\rangle = x^T y$ (standard inner product)

in $\mathbb{C}^n$: $\langle x,y\rangle = x^* y$ (standard inner product)

Proposition: $\operatorname{spec}(A^*) = \{\, \overline{\lambda} \mid \lambda \in \operatorname{spec}(A) \,\}$



The Bright Side of Mathematics

Linear Algebra – Part 60

Definition: A complex matrix $A \in \mathbb{C}^{n\times n}$ is called:

(1) selfadjoint if $A^* = A$

(2) skew-adjoint $A^* = -A$

(3) unitary if $A^*A = AA^* = \mathbb{1}$ (=identity matrix)

(4) normal if $A^*A = AA^*$

Example: (a) $A = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{1} & \overline{-2i} \\ \overline{2i} & \overline{0} \end{pmatrix} = \begin{pmatrix} 1 & 2i \\ -2i & 0 \end{pmatrix} = A$

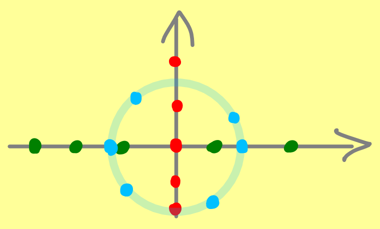
(b) $A = \begin{pmatrix} i & -1+2i \\ 1+2i & 3i \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} \overline{i} & \overline{1+2i} \\ \overline{-1+2i} & \overline{3i} \end{pmatrix} = \begin{pmatrix} -i & 1-2i \\ -1-2i & -3i \end{pmatrix} = -A$

(c) $A = \begin{pmatrix} i & 0 \\ 0 & 4 \end{pmatrix}$ not selfadoint nor skew-adjoint but normal.

Remember:

$A \in \mathbb{C}^{n\times n}$	$A \in \mathbb{R}^{n\times n}$
adjoint A^*	transpose A^T
selfadjoint	symmetric
skew-adjoint	skew-symmetric
unitary	orthogonal

Proposition: (a) A selfadjoint $\Rightarrow$ spec$(A) \subseteq$ real axis



(b) A skew-adjoint $\Rightarrow$ spec$(A) \subseteq$ imaginary axis

(c) A unitary $\Rightarrow$ spec$(A) \subseteq$ unit circle

Proof: (a) $\lambda \in$ spec$(A) \Rightarrow$ eigenvalue equation $Ax = \lambda x$, $x \neq 0$, $\boxed{\text{choose: } \|x\| = 1}$

$$\lambda \cdot \underbrace{\langle x, x\rangle}_{1} = \langle x, \lambda \cdot x\rangle = \langle x, Ax\rangle = \langle A^* x, x\rangle \overset{\text{selfadjoint}}{=} \langle Ax, x\rangle = \langle \lambda \cdot x, x\rangle = \overline{\lambda} \underbrace{\langle x, x\rangle}_{=1}$$

(c) $\lambda \in$ spec$(A) \Rightarrow$ eigenvalue equation $Ax = \lambda x$, $x \neq 0$, $\boxed{\text{choose: } \|x\| = 1}$

$$\langle \lambda x, \lambda x\rangle = \langle Ax, Ax\rangle = \langle \underbrace{A^* A}_{\mathbb{1}} x, x\rangle = \langle x, x\rangle = 1$$

$$\|$$

$$\overline{\lambda} \cdot \lambda \langle x, x\rangle = |\lambda|^2 \Rightarrow \lambda \text{ lies on the unit circle} \qquad \square$$

Linear Algebra – Part 61

Definition: $A, B \in \mathbb{C}^{n\times n}$ are called similar if there is an invertible $S \in \mathbb{C}^{n\times n}$ such that $A = S^{-1} B S$.

(For similiar matrices: f_A injective $\Leftrightarrow$ f_B injective)

(For similiar matrices: f_A surjective $\Leftrightarrow$ f_B surjective)

$\boxed{\text{change of basis}}$

Property: Similar matrices have the same characteristic polynomial.

Hence: A, B similar $\Rightarrow$ $\mathrm{spec}(A) = \mathrm{spec}(B)$

Proof:
$$p_A(\lambda) = \det(A - \lambda \mathbb{1}) = \det(S^{-1} B S - \lambda \mathbb{1}) = \det(S^{-1}(B - \lambda \mathbb{1}) S)$$
$$= \underbrace{\det(S^{-1}) \det(B - \lambda\mathbb{1}) \det(S)}_{\det(S^{-1})\det(S) = \det(\mathbb{1}) = 1} = p_B(\lambda)$$

Later:

- A normal $\Rightarrow$ $A = S^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} S$ (eigenvalues on the diagonal)
- $A \in \mathbb{C}^{n\times n}$ $\Rightarrow$ $A = S^{-1} \begin{pmatrix} \lambda_1 & & (*) \\ & \ddots & \\ & & \lambda_n \end{pmatrix} S$ (eigenvalues on the diagonal)

(Jordan normal form)

BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Linear Algebra – Part 62

Recall: $\alpha(\lambda)$ algebraic multiplicity

$\gamma(\lambda)$ geometric multiplicity $\big(=$ dimension of $\mathrm{Eig}(\lambda)\big)$

Recipe: $A \in \mathbb{C}^{n\times n}$: (1) Calculate the zeros of $p_A(\lambda) = \det(A - \lambda \mathbb{1})$.

Call them $\lambda_1, \dots, \lambda_k$,

with $\underbrace{\alpha(\lambda_1), \dots, \alpha(\lambda_k)}_{\text{sum is equal to } n}$.

$$\left[\, A \in \mathbb{R}^{n\times n}, \quad \lambda_j \text{ zero of } p_A \implies \overline{\lambda_j} \text{ zero of } p_A \,\right]$$

(2) For $j \in \{1,\dots,k\}$: Solve LES $(A - \lambda_j \mathbb{1})x = 0$

Solution set: $\mathrm{Eig}(\lambda_j)$ (eigenspace)

(3) All eigenvectors: $\displaystyle\bigcup_{j=1}^{k} \mathrm{Eig}(\lambda_j) \setminus \{0\}$

Example: $A = \begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix}$

(1) $p_A(\lambda) = \det \begin{pmatrix} 8-\lambda & 8 & 4 \\ -1 & 2-\lambda & 1 \\ -2 & -4 & -2-\lambda \end{pmatrix}$

Sarrus

$$= (8-\lambda)(2-\lambda)(-2-\lambda) + 16 - 16 + 8(2-\lambda) + 4(8-\lambda) + 8(-2-\lambda)$$

$$= (8-\lambda)(-4+\lambda^2) + 16 - 8\lambda + 32 - 4\lambda - 16 - 8\lambda$$

$$= (8-\lambda)(-4+\lambda^2) - 20\lambda + 32$$

$$= -32 + 4\lambda + 8\lambda^2 - \lambda^3 - 20\lambda + 32$$

$$= \lambda(-\lambda^2 + 8\lambda - 16) = -\lambda(\lambda-4)^2$$

$p_A(\lambda) = -\lambda^1(\lambda-4)^2$

eigenvalues:

$\lambda_1 = 0$, $\alpha(\lambda_1) = 1$

$\lambda_2 = 4$, $\alpha(\lambda_2) = 2$

(2) eigenspace for $\lambda_1 = 0$

$$\mathrm{Eig}(\lambda_1) = \mathrm{Ker}(A - \lambda_1\mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 8 & 8 & 4 \\ -1 & 2 & 1 \\ -2 & -4 & -2 \end{pmatrix} \overset{I \leftrightarrow II}{=} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 8 & 8 & 4 \\ -2 & -4 & -2 \end{pmatrix}$$

$$\underset{III + \frac{1}{3}II}{\overset{II+8I,\ III-2I}{=}} \mathrm{Ker}\begin{pmatrix} -1 & 2 & 1 \\ 0 & 24 & 12 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -\frac{1}{2}t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\} = \mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

eigenspace for $\lambda_2 = 4$

$$\mathrm{Eig}(\lambda_2) = \mathrm{Ker}(A - \lambda_2\mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 4 & 8 & 4 \\ -1 & -2 & 1 \\ -2 & -4 & -6 \end{pmatrix} \overset{I \leftrightarrow II}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 4 & 8 & 4 \\ -2 & -4 & -6 \end{pmatrix}$$

$$\overset{II+4I,\ III-2I}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix} \overset{\text{exchange, scale}}{=} \mathrm{Ker}\begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

(3) eigenvectors of A : $\left(\mathrm{Span}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \cup \mathrm{Span}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right) \setminus \{0\}$

Linear Algebra – Part 63

Assume: x eigenvector for $A \in \mathbb{C}^{n\times n}$ associated to eigenvalue $\lambda \in \mathbb{C}$

Then: $Ax = \lambda x \implies \underbrace{A(Ax)}_{= A^2 x} = A(\lambda x) = \lambda \underbrace{(Ax)}_{= \lambda x}$

$\implies A^2 x = \lambda^2 x \implies A^3 x = \lambda^3 x$

$\overset{\text{induction}}{\implies} A^m x = \lambda^m x$ for all $m \in \mathbb{N}$

Spectral mapping theorem: $A \in \mathbb{C}^{n\times n}$, $p : \mathbb{C} \to \mathbb{C}$, $p(z) = c_m z^m + \cdots + c_1 z^1 + c_0$

Define: $p(A) = c_m A^m + c_{m-1} A^{m-1} + \cdots + c_1 A + c_0 \cdot \mathbb{1}_n \in \mathbb{C}^{n\times n}$

Then: $\operatorname{spec}(p(A)) = \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$

Proof: Show two inclusion: $(\supseteq)$ (see above) ✓

$(\subseteq)$ $\boxed{\text{1st case:}}$ p constant, $p(z) = c_0$.

Take $\tilde{\lambda} \in \operatorname{spec}(p(A)) \implies \underbrace{\det(\underbrace{p(A)}_{c_0 \mathbb{1}} - \tilde{\lambda}\mathbb{1})}_{= (c_0 - \tilde{\lambda})^n} = 0$

$\implies \tilde{\lambda} \in \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$ ✓

$\boxed{\text{2nd case:}}$ p not constant. Do proof by contraposition.

Assume: $\mu \notin \{ p(\lambda) \mid \lambda \in \operatorname{spec}(A) \}$

Define polynomial: $q(z) = p(z) - \mu$

$= \underset{\neq 0}{c} \cdot (z - a_1)(z - a_2) \cdots (z - a_m)$

By definition of μ: $a_j \notin \operatorname{spec}(A)$ for all j

$\implies \det(A - a_j \mathbb{1}) \neq 0$ for all j

Hence: $\det(p(A) - \mu \mathbb{1}) = \det(q(A))$

$= \det\big(c \cdot (A - a_1)(A - a_2) \cdots (A - a_m)\big)$

$= c^n \cdot \det(A - a_1) \det(A - a_2) \cdots \det(A - a_m)$

$\neq 0$

$\implies \mu \notin \operatorname{spec}(p(A))$ □

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, $\operatorname{spec}(A) = \{1, 4\}$

$B = 3A^3 - 7A^2 + A - 2\mathbb{1}$, $\operatorname{spec}(B) = \{-5, 82\}$

The Bright Side of Mathematics

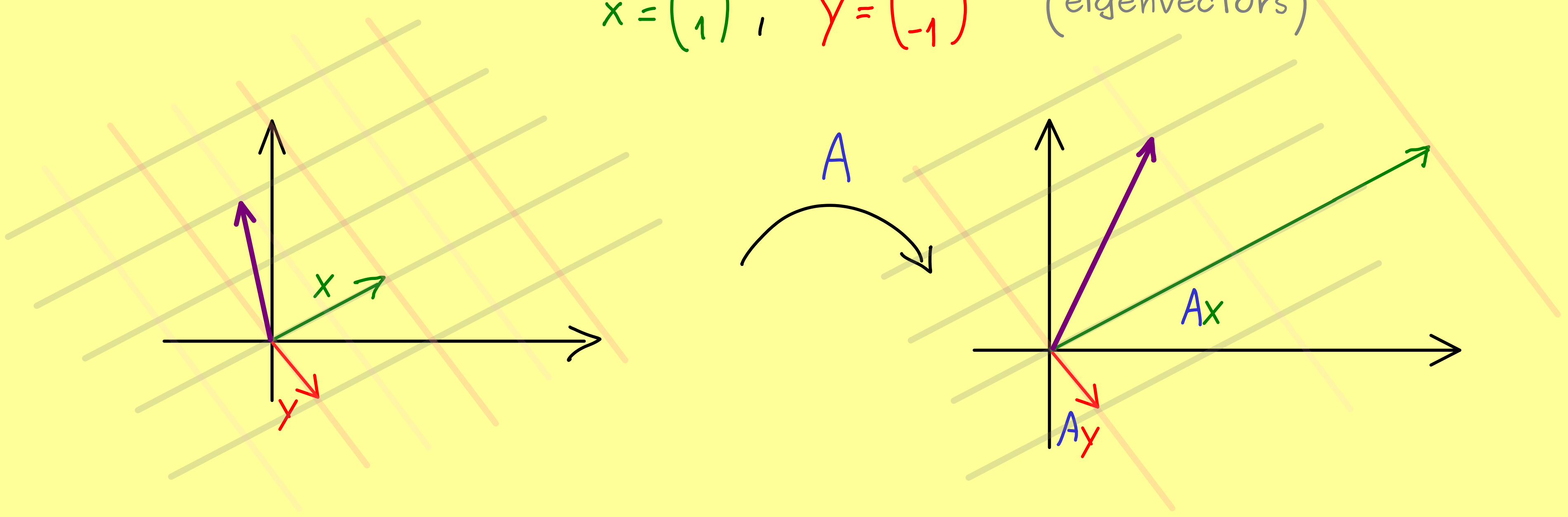
Linear Algebra – Part 64

Diagonalization = transform matrix into a diagonal one

= find a an optimal coordinate system

Example: $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, $\lambda_1 = 4$, $\lambda_2 = 1$ (eigenvalues)

$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (eigenvectors)



$$\alpha x + \beta y \longmapsto \alpha \lambda_1 x + \beta \lambda_2 y$$

Diagonalization: $A \in \mathbb{C}^{n \times n} \rightsquigarrow \lambda_1, \lambda_2, \dots, \lambda_n$ (counted with algebraic multiplicities)

$\rightsquigarrow x^{(1)}, x^{(2)}, \dots, x^{(n)}$ (associated eigenvectors)

$\rightsquigarrow A x^{(1)} = \lambda_1 x^{(1)}, \dots, A x^{(n)} = \lambda_n x^{(n)}$ (eigenvalue equations)

$$A \begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Ax^{(1)} & Ax^{(2)} & \cdots & Ax^{(n)} \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \lambda_1 x^{(1)} & \lambda_2 x^{(2)} & \cdots & \lambda_n x^{(n)} \\ | & | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \cdots & x^{(n)} \\ | & | & & | \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}}_{D}$$

$$\Longrightarrow \quad AX = XD$$

If X is invertible, then: $\boxed{D = X^{-1} A X}$ $\quad A$ is similar to a diagonal matrix

Application: $A^{98} = (X D X^{-1})^{98} = X D \underbrace{X^{-1} X}_{\mathbb{1}} D \underbrace{X^{-1} X}_{\mathbb{1}} D X^{-1} \cdots X D X^{-1}$ ...

$$= X D^{98} X^{-1}$$

$$= X \begin{pmatrix} \lambda_1^{98} & & \\ & \lambda_2^{98} & \\ & & \ddots \\ & & & \lambda_n^{98} \end{pmatrix} X^{-1}$$

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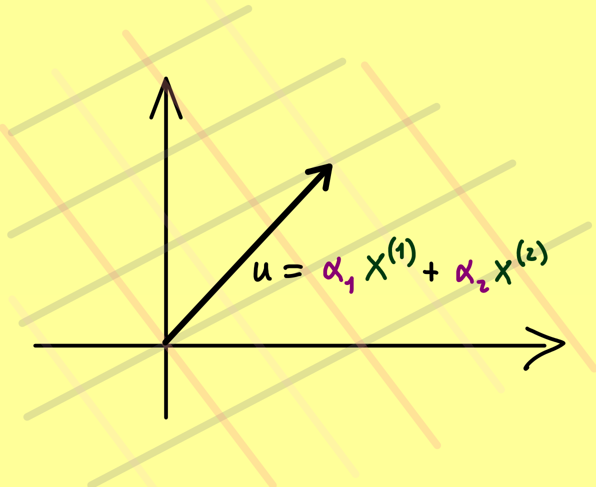
Linear Algebra – Part 65

$$
\begin{tikzcd}
\text{canonical basis:} & u \in \mathbb{C}^n \arrow[rr, bend left, "A"] \arrow[d, bend left, "X^{-1}"] & & Au \in \mathbb{C}^n \arrow[d, bend left, "X^{-1}"] \\
\text{eigenvector basis:} & \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \arrow[rr, bend left, "D"] \arrow[u, bend left, "X"] & & \begin{pmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ \vdots \\ \lambda_n \alpha_n \end{pmatrix} \arrow[u, bend left, "X"]
\end{tikzcd}
$$



$$\boxed{D = X^{-1} A X}$$

Is that possible? For given matrix $A \in \mathbb{C}^{n \times n}$ with eigenvectors $x^{(1)}, x^{(2)}, \dots, x^{(n)}$:

- Can we express each $u \in \mathbb{C}^n$ with $\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \dots + \alpha_n x^{(n)}$?
- $\mathrm{Span}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = \mathbb{C}^n$?
- $(x^{(1)}, x^{(2)}, \dots, x^{(n)})$ basis of $\mathbb{C}^n$?
- $X = \begin{pmatrix} | & | & & | \\ x^{(1)} & x^{(2)} & \dots & x^{(n)} \\ | & | & & | \end{pmatrix}$ invertible ?

Definition: $A \in \mathbb{C}^{n \times n}$ is called diagonalizable if one can find n eigenvectors of A such that they form a basis $\mathbb{C}^n$.

Example:

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, e_1, e_2 eigenvectors $\Rightarrow$ A is diagonalizable

(b) $B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvectors $\Rightarrow$ B is diagonalizable

(c) $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, all eigenvectors lie in direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ C is not diagonalizable

Remember: For $A \in \mathbb{C}^{n \times n}$:

- $\alpha(\lambda) = \gamma(\lambda)$ for all eigenvalues λ $\Leftrightarrow$ A is diagonalizable
- A normal $\Rightarrow$ A is diagonalizable
  (One can choose even an ONB with eigenvectors)
- A has n different eigenvalues $\Rightarrow$ A is diagonalizable