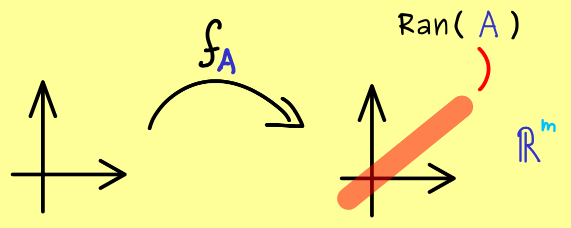
The Bright Side of Mathematics

Linear Algebra – Part 35

Definition: For $A \in \mathbb{R}^{m \times n}$ we define:



$$\operatorname{rank}(A) := \dim(\operatorname{Ran}(A))$$

$$= \dim(\text{Span of columns of } A)$$

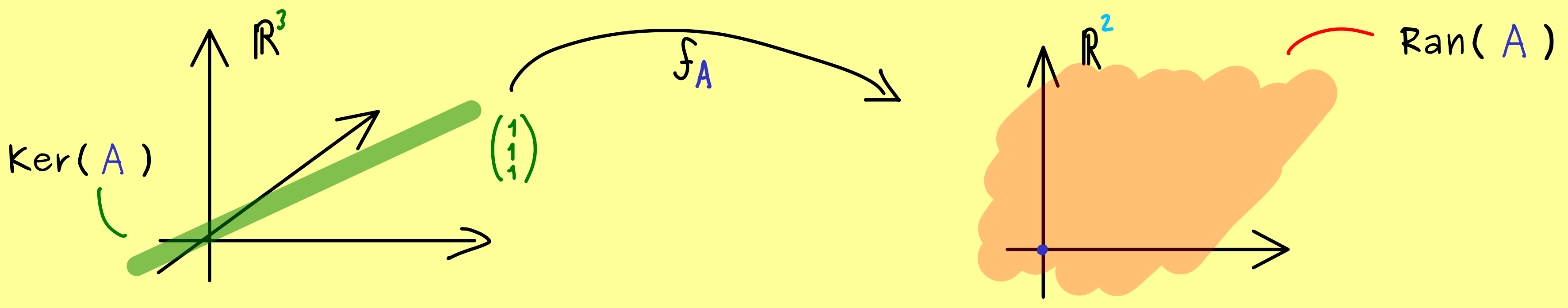
$$\leq \min(n, m)$$

A has full rank if $\operatorname{rank}(A) = \min(n, m)$

Example: (a) $A = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}$, $\operatorname{rank}(A) = 1$ (full rank)

(b) $A = \begin{pmatrix} 2 & 2 & -4 \\ 1 & 0 & -1 \end{pmatrix}$, $\operatorname{rank}(A) = 2$ (full rank)

linearly independent



Definition: For $A \in \mathbb{R}^{m \times n}$ we define:

$$\operatorname{nullity}(A) := \dim(\operatorname{Ker}(A))$$

Rank-nullity theorem: For $A \in \mathbb{R}^{m \times n}$ (n columns)

$$\dim(\operatorname{Ker}(A)) + \dim(\operatorname{Ran}(A)) = n$$

Proof: $k = \dim(\operatorname{Ker}(A))$. Choose: $(b_1, \dots, b_k)$ basis of $\operatorname{Ker}(A)$.

Steinitz Exchange Lemma $\Rightarrow$ $(b_1, \dots, b_k, c_1, \dots, c_r)$ basis of $\mathbb{R}^n$

$r := n - k$

$$\operatorname{Ran}(A) = \operatorname{Span}(\underbrace{Ab_1}_{=0}, \dots, \underbrace{Ab_k}_{=0}, Ac_1, \dots, Ac_r)$$

$$= \operatorname{Span}(Ac_1, \dots, Ac_r) \quad \Rightarrow \quad \dim(\operatorname{Ran}(A)) \leq r$$

To show: $(Ac_1, \dots, Ac_r)$ is linearly independent

$$\lambda_1 Ac_1 + \lambda_2 Ac_2 + \cdots + \lambda_r Ac_r = 0$$

$$\overset{\text{linearity}}{=} A\Big(\sum_{i=1}^{r} \lambda_i c_i\Big) \quad \Rightarrow \quad \sum_{i=1}^{r} \lambda_i c_i \in \operatorname{Ker}(A)$$

basis of kernel

$$\Rightarrow \sum_{i=1}^{r} \lambda_i c_i = \sum_{j=1}^{k} \mu_j b_j \quad \Rightarrow \quad \sum_{i=1}^{r} \lambda_i c_i + \sum_{j=1}^{k} (-\mu_j) b_j = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$$

$$\Rightarrow \dim(\operatorname{Ran}(A)) = r \qquad \square$$