Linear Algebra — Part 33

$$A \in \mathbb{R}^{m \times n} \rightsquigarrow A^T \in \mathbb{R}^{n \times m}$$

standard inner product in $\mathbb{R}^n \rightsquigarrow \langle u, v \rangle \in \mathbb{R}$

$$= u^T v$$

<u>Proposition:</u> For $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\langle y, Ax \rangle = \langle A^T y, x \rangle$$

inner product in $\mathbb{R}^m$ $\qquad$ inner product in $\mathbb{R}^n$

<u>Proof:</u> $\langle \tilde{u}, \tilde{v} \rangle = \tilde{u}^T \tilde{v}$ for $\tilde{u}, \tilde{v} \in \mathbb{R}^m$ $\qquad \left(A^T y \right)^T = y^T \left(A^T \right)^T$

$$\langle y, \widetilde{Ax} \rangle = y^T (Ax) = \left(y^T A \right) x = \left(A^T y \right)^T x = \langle A^T y, x \rangle \qquad \square$$

<u>Alternative definition:</u> A^T is the only matrix $B \in \mathbb{R}^{n \times m}$ that satisfies:

$$\langle y, Ax \rangle = \langle B y, x \rangle \qquad \text{for all } x, y$$