Exercise 1. Product metrics

In the Start Learning Logic series you already learnt about the example $(A \to B) \Leftrightarrow (\neg B \to \neg A)$ *of two logically equivalent statements. Show that the following statements are also logically equivalent by using truth tables:*

1. $\neg(A \wedge \neg B) \Leftrightarrow (A \to B)$,
2. $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$.

Exercise 2. Complement

One usually deals with subsets A, B, *etc. of a given fixed set* X. *In such a situation it is useful to introduce* $A^c := X \setminus A$ *which is called the* complement *of* A *(with respect to (w.r.t.) the set* X*). Show for* $A, B \subset X$

(a) $A \setminus B = A \cap B^c$,

(b) $(A \cap B)^c = A^c \cup B^c$.

Exercise 3. Cartesian Product

Let A, B *and* C *be sets.*

(a) Show $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) Let $|A| = n$ *and* $|B| = m$ *where* $n, m \in \mathbb{N}$. *Show that*

$$|A \times B| = n \cdot m.$$

Exercise 4. Some functions

(a) Consider the two functions $f_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ *and* $f_2 : [0, \infty) \to \mathbb{R}$, $x \mapsto x^2$. *For both functions calculate preimages of the sets* $\{1\}$, $[4, 9)$ *and* $(-1, 0)$.

(b) Consider the two functions $g_1 : \mathbb{R} \to [0, 1]$, $x \mapsto |\sin(x)|$ *and* $g_2 : [0, 2\pi] \to \mathbb{R}$, $x \mapsto \sin(x)$. *For both functions calculate images of the sets* $(0, \pi/2)$, $[0, \pi)$ *and* $(0, 2\pi]$.

(c) Consider the two functions $h_1 : \mathbb{R} \to \mathbb{R}$ *and* $h_2 : [-1, 1] \to [\sqrt{3}, 2]$ *given by*

$$x = (h_1(x) + 2)^2 - 2 \quad \text{and} \quad x^2 + h_2(x)^2 = 4.$$

Check whether h_1 *and* h_2 *respectively are correctly defined.*

(d) Consider all 6 functions from above and find out which of them are injective, surjective and bijective. Try to provide proofs and counterexamples.