The Bright Side of Mathematics

The following pages cover the whole Jordan Normal Form course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

The Bright Side of
Mathematics

Jordan normal form – part 1

A square matrix **diagonalisable** $:\Longleftrightarrow$ diagonal matrix

There is an invertible X with $X^{-1}AX = D$

$A = X\underbrace{D}_{\text{Jordan normal form}}X^{-1}$ (matrix decomposition)

$A \in \mathbb{C}^{n\times n} \Rightarrow$ There exists a Jordan normal form J for A: $A = XJX^{-1}$

Example: $A \in \mathbb{C}^{9\times 9}$ and with eigenvalues: $\{\underbrace{2,2,2}_{\substack{\text{algebraic}\\\text{multiplicity}\\3}}, \underbrace{3,3,3,3}_{\substack{\text{algebraic}\\\text{multiplicity}\\4}}, \underbrace{4,4}_{\substack{\text{algebraic}\\\text{multiplicity}\\2}}\}$

$$J = \begin{pmatrix} \boxed{\begin{matrix}2 & & (*)\\ & 2 & \\ & & 2\end{matrix}} & & \\ & \boxed{\begin{matrix}3 & & & \\ & 3 & (*) & \\ & & 3 & \\ & & & 3\end{matrix}} & \\ & & \boxed{\begin{matrix}4 & (*)\\ & 4\end{matrix}} \end{pmatrix}$$

three Jordan blocks

$\boxed{\begin{matrix}2 & & (*)\\ & 2 & \\ & & 2\end{matrix}}$:

(1) $\begin{pmatrix}\boxed{2} & & \\ & \boxed{2} & \\ & & \boxed{2}\end{pmatrix}$ three Jordan boxes , geometric multiplicity : 3

(2) $\begin{pmatrix}\boxed{\begin{matrix}2 & 1\\ & 2\end{matrix}} & \\ & \boxed{2}\end{pmatrix}$ two Jordan boxes , geometric multiplicity : 2

(3) $\boxed{\begin{matrix}2 & 1 & \\ & 2 & 1\\ & & 2\end{matrix}}$ one Jordan boxes , geometric multiplicity : 1

$\boxed{\begin{matrix}3 & & & \\ & 3 & (*) & \\ & & 3 & \\ & & & 3\end{matrix}}$:

(1) $\begin{pmatrix}\boxed{3} & & & \\ & \boxed{3} & & \\ & & \boxed{3} & \\ & & & \boxed{3}\end{pmatrix}$ four Jordan boxes , geometric multiplicity 4

(2) $\begin{pmatrix}\boxed{\begin{matrix}3 & 1\\ & 3\end{matrix}} & & \\ & \boxed{3} & \\ & & \boxed{3}\end{pmatrix}$ three Jordan boxes , geometric multiplicity 3

(3) $\begin{pmatrix}\boxed{\begin{matrix}3 & 1\\ & 3\end{matrix}} & \\ & \boxed{\begin{matrix}3 & 1\\ & 3\end{matrix}}\end{pmatrix}$ (4) $\begin{pmatrix}\boxed{\begin{matrix}3 & 1 & \\ & 3 & 1\\ & & 3\end{matrix}} & \\ & \boxed{3}\end{pmatrix}$ two Jordan boxes
geometric multiplicity 2

(5) $\boxed{\begin{matrix}3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1\\ & & & 3\end{matrix}}$ one Jordan boxes , geometric multiplicity 1

$\boxed{XJX^{-1} = A}$

Recipe: (1) Calculate eigenvalues of A : $\lambda_1, \lambda_2, \dots, \lambda_k$

For $i = 1..k$: (2) Determine algebraic multiplicity of λ_i and geometric multiplicity of $\lambda_i := \dim(\mathrm{Ker}(A - \lambda_i \mathbb{1}))$

(3) If needed: $\dim(\mathrm{Ker}(A-\lambda_i\mathbb{1})^2)$, $\dim(\mathrm{Ker}(A-\lambda_i)^3)$ ...

Jordan normal form – part 2

$$A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ -1 & 5 & 4 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Find a Jordan normal form for the matrix A.

Recipe:

(1) Eigenvalues of A

(2) Algebraic and geometric multiplicities

(3) Dimensions of generalised eigenspaces

(1) Eigenvalues:
$$\det(A - \lambda \cdot \mathbb{1}) = \det \begin{pmatrix} 3-\lambda & 1 & 0 & 1 \\ -1 & 5-\lambda & 4 & 1 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{pmatrix}$$

$$= \det \begin{pmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{pmatrix} \cdot \det \begin{pmatrix} 2-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$$

$$= \big((3-\lambda)(5-\lambda) + 1 \big) \cdot (2-\lambda)(4-\lambda)$$

$$= (16 - 8\lambda + \lambda^2) \cdot (2-\lambda) \cdot (4-\lambda)$$

$$= (4-\lambda)^2 \cdot (2-\lambda) \cdot (4-\lambda) = (2-\lambda)^1 \cdot (4-\lambda)^3$$

Eigenvalues $\lambda_1 = 2$ with $\alpha(\lambda_1) = 1$ ⇝ Jordan block 1×1

$\lambda_2 = 4$ with $\alpha(\lambda_2) = 3$ ⇝ Jordan block 3×3

$$J = \begin{pmatrix} \boxed{2} & & & \\ & 4 & 1 & \\ & & 4 & \\ & & & 4 \end{pmatrix}$$

1st possibility: $\begin{pmatrix} 4 & & \\ & 4 & \\ & & 4 \end{pmatrix}$ $\quad \gamma(\lambda_2) = 3$ (geometric multiplicity)

2nd possibility: $\begin{pmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{pmatrix}$ $\quad \gamma(\lambda_2) = 2$

3rd possibility: $\begin{pmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{pmatrix}$ $\quad \gamma(\lambda_2) = 1$

Eigenspace for $\lambda_2 = 4$:

$$\operatorname{Ker}(A - \lambda_2 \mathbb{1}) = \operatorname{Ker} \begin{pmatrix} 3-4 & 1 & 0 & 1 \\ -1 & 5-4 & 4 & 1 \\ 0 & 0 & 2-4 & 0 \\ 0 & 0 & 0 & 4-4 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} \boxed{-1} & 1 & 0 & 1 \\ -1 & 1 & 4 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\mathrm{II} - \mathrm{I}}{=} \operatorname{Ker} \begin{pmatrix} \boxed{-1} & 1 & 0 & 1 \\ 0 & 0 & \boxed{4} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\mathrm{III} + \frac{1}{2}\mathrm{II}}{=} \operatorname{Ker} \begin{pmatrix} \boxed{-1} & 1 & 0 & 1 \\ 0 & 0 & \boxed{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_4 free variables

$$\gamma(\lambda_2) = \dim\big(\operatorname{Ker}(A - \lambda_2 \mathbb{1})\big) = 2 \Leftarrow$$

Jordan normal form – part 3

$$A = \begin{pmatrix} 2 & 1 & -1 & 8 & -3 \\ 0 & 2 & 0 & 7 & 5 \\ 0 & 0 & 2 & 7 & 5 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Recipe: (1) Eigenvalues
(2) Alg. and geom. multiplicities
(3) Dimensions of the generalised eigenspaces

Find a Jordan normal form for A:

(1) Eigenvalues: $\det(A - \lambda \mathbb{1}) = (2-\lambda)^5 \Rightarrow \lambda_1 = 2$

For $\lambda_1 = 2$: (2) Multiplicities: $\alpha(\lambda_1) = 5$

3 free variables

$$\mathrm{Ker}(A - \lambda_1 \mathbb{1}) = \mathrm{Ker}\begin{pmatrix} 0 & 1 & -1 & 8 & -3 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\text{III}-\text{II}}{=} \mathrm{Ker}\begin{pmatrix} 0 & \boxed{1} & -1 & 8 & -3 \\ 0 & 0 & 0 & \boxed{7} & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \gamma(\lambda_1) = 3$ (three boxes!)

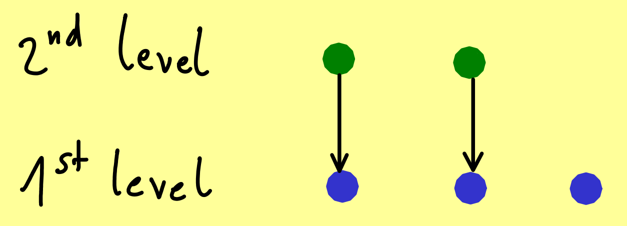
Possibility one: $\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}$

Possibility two: $\begin{pmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 2 & 1 & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}$

(3) Generalised eigenspaces:

$$\mathrm{Ker}(A - \lambda_1 \mathbb{1})^2 = \mathrm{Ker}\left(\begin{pmatrix} 0 & 1 & -1 & 8 & -3 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 & 8 & -3 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

$$= \mathrm{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim\left(\mathrm{Ker}(A - \lambda_1 \mathbb{1})^2\right) = 5$$



$$\Rightarrow J = \begin{pmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 2 & 1 & \\ & & & 2 & \\ & & & & 2 \end{pmatrix}$$

Jordan normal form part 4

$A \in \mathbb{C}^{n \times n} \Rightarrow A = X J X^{-1}$

X: transformation matrix, J: Jordan normal form

Recipe:
(1) Eigenvalues
(2) Alg. and geom. multiplicities
(3) Calculate the eigenspace and generalised eigenspaces
(4) transformation matrix X

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \text{Find } J \text{ and } X!$$

(1) Eigenvalues: $\det(A - \lambda \mathbb{1}) = (1-\lambda)^5 \Rightarrow \lambda_1 = 1$

(2) Multiplicities: For $\lambda_1 = 1$: algebraic multiplicity $\alpha(\lambda_1) = 5 \rightsquigarrow$ Jordan block of size 5x5

$$\operatorname{Ker}(A - \lambda_1 \mathbb{1}) = \operatorname{Ker}\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{\text{Gaussian elimin.}}{=} \operatorname{Ker}\begin{pmatrix} 0 & \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & \boxed{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

geometric multiplicity $\gamma(\lambda_1) = 2 \rightsquigarrow$ two Jordan boxes

(3) Eigenspace and generalised eigenspaces: For $\lambda_1 = 1$:

$$\operatorname{Ker}(A - \lambda_1 \mathbb{1}) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right) \rightsquigarrow \text{two linearly independent eigenvectors}$$

$$\underbrace{\operatorname{Ker}(A - \lambda_1 \mathbb{1})^2}_{\text{4-dimensional}} = \operatorname{Ker}\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^2 = \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\text{Gaussian elimin.}}{=} \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$\operatorname{Ker}(A - \lambda_1 \mathbb{1})^3 = \operatorname{Ker}\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^3 = \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

3rd level w_3 ← generalised eigenvectors of level 3
2nd level w_2 u_2 ← generalised eigenvectors of level 2
1st level w_1 u_1 ← eigenvectors

Choose w_3: generalised eigenvectors of level 3 : $w_3 \in \operatorname{Ker}(A - \lambda_1 \mathbb{1})^3$ but $w_3 \notin \operatorname{Ker}(A - \lambda_1 \mathbb{1})^2$.

Then set: $\boxed{w_{k-1} := (A - \lambda_1 \mathbb{1}) w_k}$

$$w_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Choose u_2: generalised eigenvectors of level 2 :

$$u_2 \in \operatorname{Ker}(A - \lambda_1 \mathbb{1})^2 \text{ but } u_2 \notin \underbrace{\operatorname{Span}\left(\operatorname{Ker}(A - \lambda_1 \mathbb{1})^1 \cup \{w_2\}\right)}_{\operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)}$$

$$\left[\operatorname{Ker}(A - \lambda_1 \mathbb{1}) = \operatorname{Span}\left(\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{w_1} \right), \quad \operatorname{Ker}(A - \lambda_1 \mathbb{1})^2 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right), \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then set: $$u_1 = (A - \lambda_1 \mathbb{1}) u_2 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(4) transformation matrix:

$$X = \Bigg(\underbrace{\underbrace{w_1 \;\; w_2 \;\; w_3}_{\text{Jordan box}} \;\; \underbrace{u_1 \;\; u_2}_{\text{Jordan box}}}_{\lambda_1 = 1 \text{ Jordan block}} \Bigg) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$(XJ = AX)$

$$\rightsquigarrow J = X^{-1} A X = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$