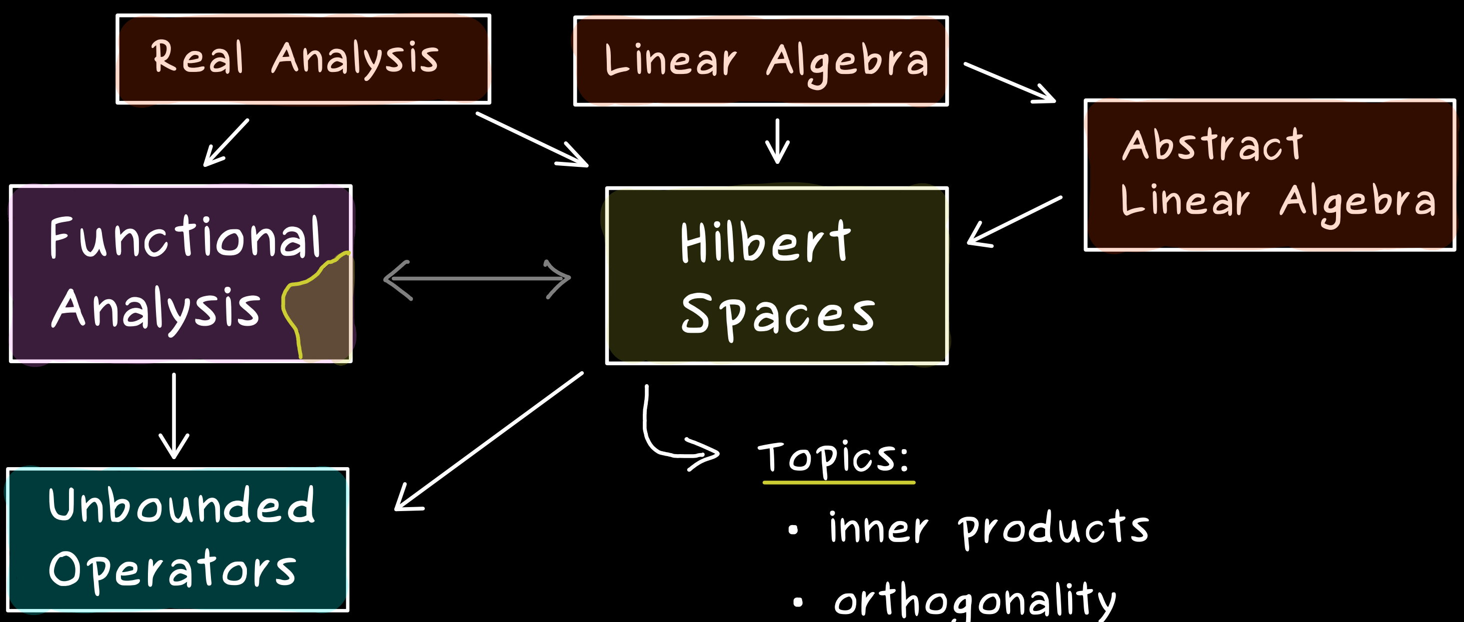
The Bright Side of Mathematics

The following pages cover the whole Hilbert Spaces course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Hilbert Spaces - Part 1



Topics:

- inner products
- orthogonality
- orthonormal basis
- orthogonal projections
- unitary + self-adjoint operators

Definition: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. An $\mathbb{F}$- vector space X with inner product $\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{F}$, which means

(1) $\langle x , x \rangle \geq 0$ for all $x \in X$ (positive definite)

and $\langle x , x \rangle = 0 \implies x = 0$ (zero vector)

(2) $\langle y , x + \tilde{x} \rangle = \langle y , x \rangle + \langle y , \tilde{x} \rangle$ for all $x, \tilde{x}, y \in X$

$\langle y , \lambda \cdot x \rangle = \lambda \cdot \langle y , x \rangle$ for all $\lambda \in \mathbb{F}, x, \tilde{x}, y \in X$

(linear in the second argument)

(3) $\langle x , y \rangle = \overline{\langle y , x \rangle}$ for all $x, y \in X$ (conjugate symmetric)

is called an inner-product space. (pre-Hilbert space)

<u>Cauchy-Schwarz inequality:</u> For an inner product space $(X, \langle\cdot,\cdot\rangle)$, we have:

$$|\langle y, x\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle \quad \text{for all} \quad x, y \in X$$

<u>Proof:</u> For $y \neq 0$:

$$0 \leq \left\langle x - \frac{\langle y,x\rangle}{\langle y,y\rangle} \cdot y \,,\, x - \frac{\langle y,x\rangle}{\langle y,y\rangle} \cdot y \right\rangle$$

$$= \langle x,x\rangle - \cancel{\frac{\overline{\langle y,x\rangle}}{\langle y,y\rangle} \cdot \langle y,x\rangle} - \frac{\langle y,x\rangle}{\langle y,y\rangle} \cdot \langle x,y\rangle$$

$$+ \cancel{\frac{\overline{\langle y,x\rangle}}{\langle y,y\rangle} \cdot \frac{\langle y,x\rangle}{\langle y,y\rangle}} \cdot \frac{\langle y,y\rangle}{}$$

$$= \langle x,x\rangle - \frac{|\langle y,x\rangle|^2}{\langle y,y\rangle} \qquad \square$$

<u>Result:</u> $\|x\| := \sqrt{\langle x,x\rangle}$ defines a <u>norm</u> on X

<u>Definition:</u> An inner product space $(X, \langle\cdot,\cdot\rangle)$ is called a <u>Hilbert space</u> if $(X, \|\cdot\|)$ is complete.

Hilbert Spaces - Part 2

Definition (Hilbert space): $(X, \langle \cdot , \cdot \rangle)$ $\mathbb{F}$ - vector space

$\langle \cdot , \cdot \rangle : X \times X \longrightarrow \mathbb{F}$ inner product

where $(X, \|\cdot\|)$ is a Banach space

with respect to the norm $\|x\| := \sqrt{\langle x, x \rangle}$

Example:

(a) $\mathbb{C}^N$ with standard inner product

(b) $\mathbb{R}^n$ with given inner product

(finite-dimensional normed vector spaces are always complete)

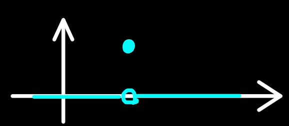
(c) $\ell^2(\mathbb{N}, \mathbb{C}) := \Big\{ \underbrace{(x_n)_{n \in \mathbb{N}}}_{= x} \;\Big|\; x_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty \Big\}$

with inner product: $\langle y, x \rangle = \sum_{n=1}^{\infty} \overline{y_n} \cdot x_n$ (convergent series!)

(d) $(\Omega, \mathcal{A}, \mu)$ measure space

$$\mathcal{L}^2(\Omega, \mu) := \Big\{ f : \Omega \longrightarrow \mathbb{C} \text{ measurable} \;\Big|\; \int_\Omega |f|^2 \, d\mu < \infty \Big\}$$

$\|f\| := \sqrt{\int_\Omega |f|^2 \, d\mu}$ not a norm in general!



$$L^2(\Omega, \mu) := \mathcal{L}^2(\Omega, \mu) \big/ \mathcal{N} \quad \text{where } \mathcal{N} := \Big\{ \underset{\text{measurable}}{f : \Omega \to \mathbb{C}} \;\Big|\; \|f\| = 0 \Big\}$$

$\| [f] \| := \|f\|$ well-defined $\rightsquigarrow$ norm on $L^2(\Omega, \mu)$

We get a Hilbert space with the following inner product:

$$\langle [g], [f] \rangle := \int_\Omega \overline{g(\omega)} \, f(\omega) \, d\mu(\omega)$$

Hilbert Spaces - Part 3

$(X, \langle \cdot, \cdot \rangle)$ inner product space ($\mathbb{F}$ - vector space + inner product)

$\Longrightarrow$ $(X, \|\cdot\|)$ normed space with $\|x\| := \sqrt{\langle x, x\rangle}$

norm induced by inner product

Polarization identity: (for case $\mathbb{F} = \mathbb{C}$)

$(X, \langle \cdot, \cdot \rangle)$ inner product space with induced norm $\|\cdot\|$. Then, for all $x, y \in X$:

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\,\|x+iy\|^2 + i\,\|x-iy\|^2 \right)$$

inner product is linear in the second argument

Proof:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$

$$-\|x-y\|^2 = -\langle x-y, x-y\rangle = -\langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle - \langle y,y\rangle$$

$$-i\cdot\|x+iy\|^2 = -i\langle x+iy, x+iy\rangle = -i\langle x,x\rangle - \langle y,x\rangle + \langle x,y\rangle - i\langle y,y\rangle$$

$$i\,\|x-iy\|^2 = i\langle x-iy, x-iy\rangle = i\langle x,x\rangle - \langle y,x\rangle + \langle x,y\rangle + i\langle y,y\rangle$$

$\square$

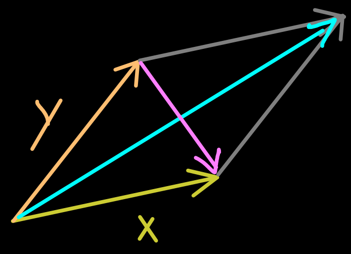
Polarization identity: (for case $\mathbb{F} = \mathbb{R}$)

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right) \quad \text{for all } x, y \in X.$$

Hilbert Spaces - Part 4

$(X, \langle \cdot,\cdot \rangle)$ inner product space ($\mathbb{F}$ - vector space + inner product)

$$\|x\|_{\langle\cdot,\cdot\rangle} := \sqrt{\langle x, x\rangle} \quad \text{induced norm}$$

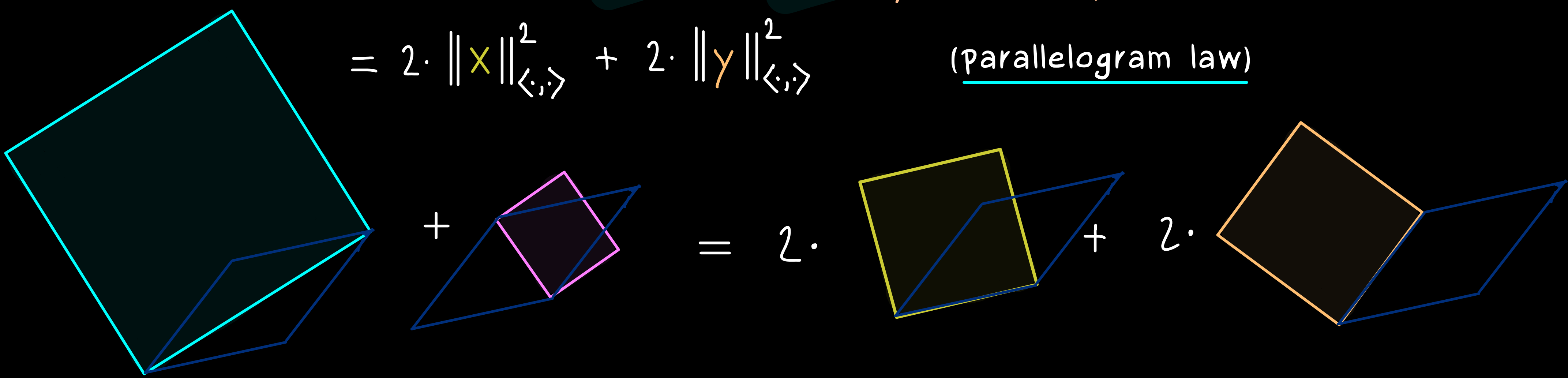


We get: $\|x+y\|^2_{\langle\cdot,\cdot\rangle} + \|x-y\|^2_{\langle\cdot,\cdot\rangle}$

$$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$+ \langle x,x\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle y,y\rangle$$

$$= 2\cdot \|x\|^2_{\langle\cdot,\cdot\rangle} + 2\cdot \|y\|^2_{\langle\cdot,\cdot\rangle} \qquad \text{(parallelogram law)}$$



$$\|x+y\|^2_{\langle\cdot,\cdot\rangle} + \|x-y\|^2_{\langle\cdot,\cdot\rangle} = 2\cdot \|x\|^2_{\langle\cdot,\cdot\rangle} + 2\cdot \|y\|^2_{\langle\cdot,\cdot\rangle}$$

Proposition: Let $(X, \|\cdot\|)$ be a normed space. Then:

the parallelogram law is satisfied $\left(\forall x,y\in X: \|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2\right)$

$\Longleftrightarrow$ $\|\cdot\|$ is induced by an inner product on X $\left(\|\cdot\|_{\langle\cdot,\cdot\rangle} = \|\cdot\|\right)$

next video

In this case: $\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ for $\mathbb{F} = \mathbb{R}$

$\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\,\|x+iy\|^2 + i\,\|x-iy\|^2\right)$ for $\mathbb{F} = \mathbb{C}$

gives the inner product on X.

Remember: A Hilbert space is a Banach space where the parallelogram law holds.

Hilbert Spaces - Part 5

Jordan-von Neumann Theorem: Let $(X, \|\cdot\|)$ be a normed space. Then:

the parallelogram law is satisfied $\left(\forall x,y \in X: \|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2\right)$

$\Longrightarrow$ $\|\cdot\|$ is induced by an inner product on X

(there is an inner product $\langle\cdot,\cdot\rangle$ on X such that $\|x\| := \sqrt{\langle x,x\rangle}$)

In this case: $\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ for $\mathbb{F}=\mathbb{R}$

$\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\,\|x+iy\|^2 + i\,\|x-iy\|^2\right)$

gives the inner product on X. for $\mathbb{F}=\mathbb{C}$

Proof: Consider case $\mathbb{F}=\mathbb{R}$. So we define: $\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$.

To show three properties:
(1) positive definite
(2) linear in the second argument
(3) symmetry

(1): $\langle x,x\rangle = \frac{1}{4}\left(\|x+x\|^2 - \|x-x\|^2\right) = \frac{1}{4}\|2\cdot x\|^2 = \|x\|^2 \geq 0$

and $\langle x,x\rangle = 0 \Longrightarrow x = 0$

(3): $\langle y,x\rangle = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2\right) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \langle x,y\rangle$

(2) linearity: $\boxed{\text{we will use: } \|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2}$

First step: $\langle w, z\rangle = \frac{1}{4}\left(\|w+z\|^2 - \|w-z\|^2\right)$

$$= \frac{1}{4}\Big(\|\underbrace{w+z}_{\overset{\shortparallel}{x+y}}\|^2 + \|\underbrace{w}_{\overset{\shortparallel}{x-y}}\|^2 - \big(\|\underbrace{w}_{\overset{\shortparallel}{\tilde{x}+\tilde{y}}}\|^2 + \|\underbrace{w-z}_{\overset{\shortparallel}{\tilde{x}-\tilde{y}}}\|^2\big)\Big)$$

$x := w + \frac{1}{2}z$ $\quad$ $\tilde{x} := w - \frac{1}{2}z$

$y := \frac{1}{2}z$ $\quad$ $\tilde{y} := \frac{1}{2}z$

parallelogram law

$$= \frac{1}{4}\left(2\cdot\|x\|^2 + 2\cdot\|y\|^2 - \left(2\cdot\|\tilde{x}\|^2 + 2\cdot\|\tilde{y}\|^2\right)\right)$$

$$= \frac{1}{2}\left(\|x\|^2 - \|\tilde{x}\|^2\right) = \frac{1}{2}\left(\|w+\tfrac{1}{2}z\|^2 - \|w-\tfrac{1}{2}z\|^2\right)$$

$$= 2\cdot\langle w, \tfrac{1}{2}z\rangle$$

First result: $\frac{1}{2}\langle w, z\rangle = \langle w, \frac{1}{2}z\rangle \xrightarrow[n\in\mathbb{N}]{\text{induction}} \boxed{\frac{1}{2^n}\langle w, z\rangle = \langle w, \frac{1}{2^n}z\rangle \quad (*)}$

Additivity: $\langle w, z\rangle + \langle w, \hat{z}\rangle$

$$= \frac{1}{4}\left(\|w+z\|^2 - \|w-z\|^2\right) + \frac{1}{4}\left(\|w+\hat{z}\|^2 - \|w-\hat{z}\|^2\right)$$

$$= \frac{1}{4}\Big(\|w + \tfrac{z+\hat{z}}{2} + \tfrac{z-\hat{z}}{2}\|^2 + \|w + \tfrac{z+\hat{z}}{2} - \tfrac{z-\hat{z}}{2}\|^2$$
$$- \left(\|w - \tfrac{z+\hat{z}}{2} + \tfrac{z-\hat{z}}{2}\|^2 + \|w - \tfrac{z+\hat{z}}{2} - \tfrac{z-\hat{z}}{2}\|^2\right)\Big)$$

parallelogram law

$$= \frac{1}{4}\left(2\cdot\|w + \tfrac{z+\hat{z}}{2}\|^2 + 2\,\|\tfrac{z-\hat{z}}{2}\|^2 - \left(2\,\|w - \tfrac{z+\hat{z}}{2}\|^2 + 2\,\|\tfrac{z-\hat{z}}{2}\|^2\right)\right)$$

$$= \frac{1}{2}\left(\|w + \tfrac{z+\hat{z}}{2}\|^2 - \|w - \tfrac{z+\hat{z}}{2}\|^2\right) = 2\,\langle w, \tfrac{z+\hat{z}}{2}\rangle$$

$$\overset{(*)}{=} \langle w, z+\hat{z}\rangle$$

Homogeneity:

$$\langle w, z\rangle + \langle w, z\rangle \overset{\text{additivity}}{=} \langle w, z+z\rangle$$

$$\langle w, z\rangle + \langle w, z\rangle = 2\cdot\langle w, z\rangle, \qquad \langle w, z+z\rangle = \langle w, 2\cdot z\rangle$$

$$\xrightarrow[k\in\mathbb{N}]{\text{induction}} \quad k\cdot\langle w, z\rangle = \langle w, k z\rangle$$

combining with $(*)$: $\frac{k}{2^n}\langle w, z\rangle = \langle w, \frac{k}{2^n} z\rangle$ for all $k, n \in \mathbb{N}$

$$0\cdot\langle w, z\rangle = \langle w, 0\cdot z\rangle$$

$$(-1)\cdot\langle w, z\rangle = \langle w, (-1)\cdot z\rangle$$

all positive
real numbers
can be approximated

$\square$

Hilbert Spaces - Part 6

$$(X, \underbrace{\langle \cdot , \cdot \rangle})$$

$\rightarrow$ gives geometry to vector space X

$\rightarrow$ we can measure lengths: $\|x\| := \sqrt{\langle x, x \rangle}$

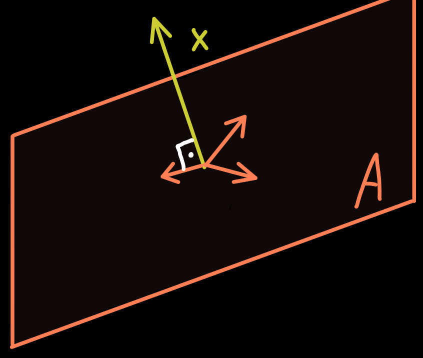
$\rightarrow$ we can measure angles / orthogonality

Definition: $(X, \langle \cdot , \cdot \rangle)$ inner product space.

(1) $x \in X$ is orthogonal to $y \in X$ if $\langle x, y \rangle = 0$. Write $x \perp y$.

(2) $x \in X$ is called orthogonal to $A \subseteq X$ if $\langle x, a \rangle = 0$ for all $a \in A$.

We write $x \perp A$.

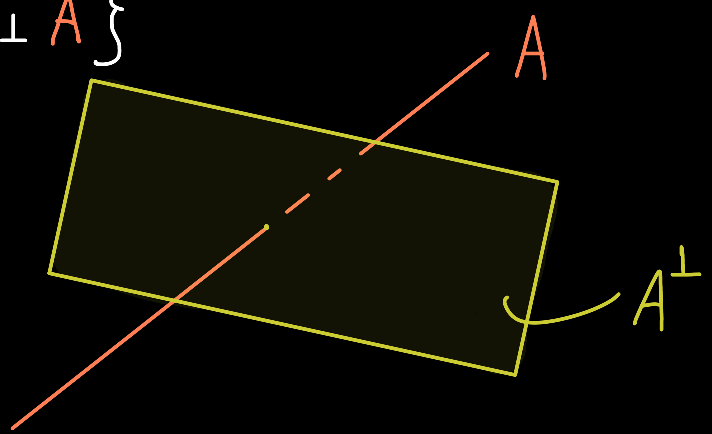


(3) $B \subseteq X$ is called orthogonal to $A \subseteq X$ if $\langle b, a \rangle = 0$ for all $a \in A$ for all $b \in B$

We write $B \perp A$.

(4) The orthogonal complement of $A \subseteq X$ is defined by:

$$A^{\perp} := \{ x \in X \mid x \perp A \}$$



Properties: $(X, \langle \cdot, \cdot \rangle)$ inner product space, $A \subseteq X$.

(a) $A^\perp$ is a subspace in X.

(b) $A^\perp$ is closed in X (complement $X \setminus A^\perp$ is an open set)

(c) $A^\perp = \overline{A}^\perp$

(d) $A^\perp = \mathrm{Span}(A)^\perp$

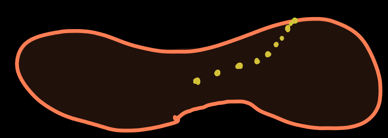
Proof: (a) $x, y \in A^\perp$, $a \in A$, $\lambda \in \mathbb{F}$

$$\Rightarrow \langle x+y, a \rangle = \langle x, a \rangle + \langle y, a \rangle = 0$$

$$\langle 0, a \rangle = 0$$

$$\langle \lambda \cdot x, a \rangle = \overline{\lambda} \cdot \langle x, a \rangle = 0 \qquad \Rightarrow A^\perp \text{ subspace in } X.$$

(b) Take $(x_n)_{n \in \mathbb{N}} \subseteq A^\perp$ with $x_n \xrightarrow{n \to \infty} x \in X$.



$A^\perp$

For any $a \in A$:

inner product continuous
in both arguments

$$0 = \lim_{n \to \infty} \langle x_n, a \rangle = \langle \lim_{n \to \infty} x_n, a \rangle = \langle x, a \rangle \Rightarrow x \in A^\perp$$

(c) $A \subseteq \overline{A} \Rightarrow A^\perp \supseteq \overline{A}^\perp$

Other inclusion? $(\subseteq)$ $x \in A^\perp$, $b \in \overline{A}$, choose $(a_n) \subseteq A$ with $\lim_{n \to \infty} a_n = b$

$$\langle x, b \rangle = \langle x, \lim_{n \to \infty} a_n \rangle = \lim_{n \to \infty} \langle x, a_n \rangle = 0$$

inner product continuous
in both arguments

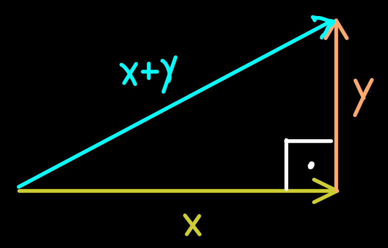
$$\Rightarrow x \in \overline{A}^\perp$$

(d) $A \subseteq \mathrm{Span}(A) \Rightarrow A^\perp \supseteq \mathrm{Span}(A)^\perp$

Other inclusion? $(\subseteq)$ $x \in A^\perp$, $\sum_{j=1}^{n} \lambda_j \cdot a_j \in \mathrm{Span}(A)$:

$$\langle x, \sum_{j=1}^{n} \lambda_j \cdot a_j \rangle = \sum_{j=1}^{n} \lambda_j \cdot \langle x, a_j \rangle = 0 \Rightarrow x \in \mathrm{Span}(A)^\perp$$

Hilbert Spaces - Part 7



choose x, y orthogonal: $\langle x, y \rangle = 0$

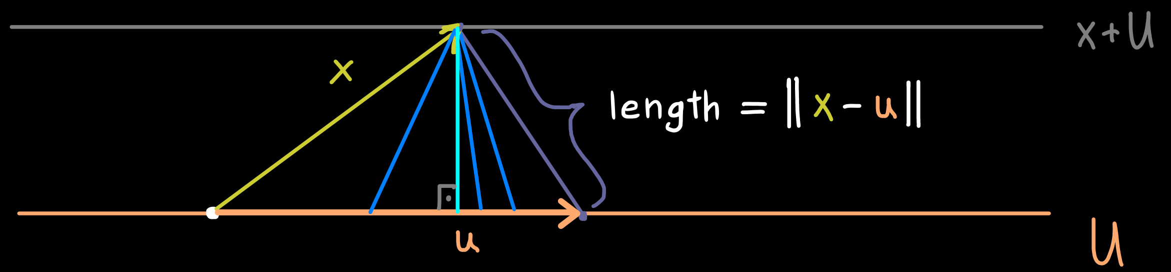
Pythagorean theorem: $(X, \langle \cdot, \cdot \rangle)$ inner product space with induced norm $\|\cdot\|$.

For any $x, y \in X$ with $x \perp y$, we have:

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \underbrace{\langle y, x \rangle}_{=0} + \underbrace{\langle x, y \rangle}_{=0} + \langle y, y \rangle$$

$$= \|x\|^2 + \|y\|^2$$

Approximation Formula

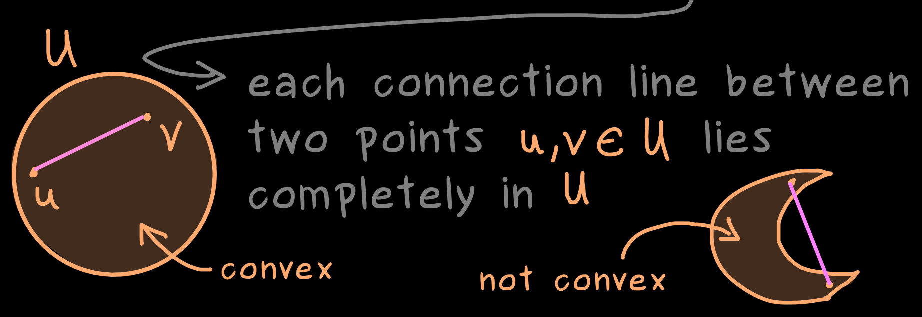


distance between $x+U$ and U: $\inf \{ \|x-u\| \mid u \in U \} =: \mathrm{dist}(x, U)$

Theorem: Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $U \subseteq X$ be closed and convex.

For every $x \in X$ there exists a unique best approximation:

$$x|_U \in U$$



This means: $\|x - x|_U\| = \mathrm{dist}(x, U)$