Hilbert Spaces - Part 6

$(X, \langle\cdot,\cdot\rangle)$

$\langle\cdot,\cdot\rangle$ → gives geometry to vector space X

→ we can measure lengths: $\|x\| := \sqrt{\langle x, x\rangle}$

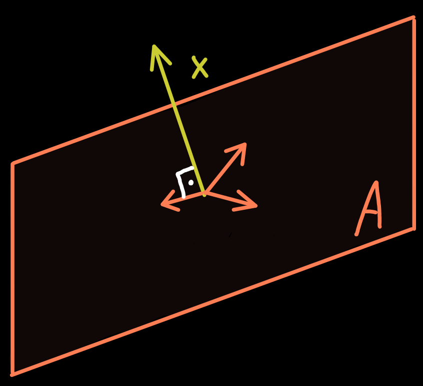
→ we can measure angles / orthogonality

Definition: $(X, \langle\cdot,\cdot\rangle)$ inner product space.

(1) $x \in X$ is orthogonal to $y \in X$ if $\langle x, y\rangle = 0$. Write $x \perp y$.

(2) $x \in X$ is called orthogonal to $A \subseteq X$ if $\langle x, a\rangle = 0$ for all $a \in A$.

We write $x \perp A$.

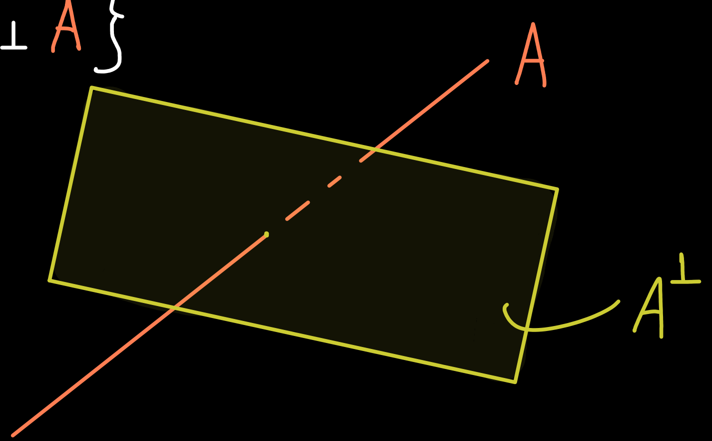


(3) $B \subseteq X$ is called orthogonal to $A \subseteq X$ if $\langle b, a\rangle = 0$ for all $a \in A$ for all $b \in B$

We write $B \perp A$.

(4) The orthogonal complement of $A \subseteq X$ is defined by:

$$A^{\perp} := \{ x \in X \mid x \perp A \}$$



Properties: $(X, \langle \cdot, \cdot \rangle)$ inner product space, $A \subseteq X$.

(a) $A^\perp$ is a subspace in X.

(b) $A^\perp$ is closed in X (complement $X \setminus A^\perp$ is an open set)

(c) $A^\perp = \overline{A}^\perp$

(d) $A^\perp = \operatorname{Span}(A)^\perp$

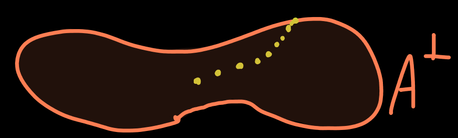
Proof: (a) $x, y \in A^\perp$, $a \in A$, $\lambda \in \mathbb{F}$

$$\Rightarrow \langle x + y, a \rangle = \langle x, a \rangle + \langle y, a \rangle = 0$$

$$\langle 0, a \rangle = 0$$

$$\langle \lambda \cdot x, a \rangle = \overline{\lambda} \cdot \langle x, a \rangle = 0 \qquad \Rightarrow \quad A^\perp \text{ subspace in } X.$$

(b) Take $(x_n)_{n \in \mathbb{N}} \subseteq A^\perp$ with $x_n \xrightarrow{n \to \infty} x \in X$.



For any $a \in A$:

inner product continuous in both arguments

$$0 = \lim_{n \to \infty} \langle x_n, a \rangle = \langle \lim_{n \to \infty} x_n, a \rangle = \langle x, a \rangle \quad \Rightarrow \quad x \in A^\perp$$

(c) $A \subseteq \overline{A} \quad \Rightarrow \quad A^\perp \supseteq \overline{A}^\perp$

Other inclusion? $(\subseteq)$ $x \in A^\perp$, $b \in \overline{A}$, choose $(a_n) \subseteq A$ with $\lim_{n \to \infty} a_n = b$

$$\langle x, b \rangle = \langle x, \lim_{n \to \infty} a_n \rangle = \lim_{n \to \infty} \langle x, a_n \rangle = 0$$

inner product continuous in both arguments

$$\Rightarrow \quad x \in \overline{A}^\perp$$

(d) $A \subseteq \operatorname{Span}(A) \quad \Rightarrow \quad A^\perp \supseteq \operatorname{Span}(A)^\perp$

Other inclusion? $(\subseteq)$ $x \in A^\perp$, $\sum_{j=1}^{n} \lambda_j \cdot a_j \in \operatorname{Span}(A)$:

$$\langle x, \sum_{j=1}^{n} \lambda_j \cdot a_j \rangle = \sum_{j=1}^{n} \overline{\lambda_j} \cdot \langle x, a_j \rangle = 0 \quad \Rightarrow \quad x \in \operatorname{Span}(A)^\perp$$