Hilbert Spaces - Part 5

Jordan-von Neumann Theorem: Let $(X, \|\cdot\|)$ be a normed space. Then:

the parallelogram law is satisfied $\left(\forall x,y \in X: \|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2\right)$

$\Longrightarrow$ $\|\cdot\|$ is induced by an inner product on X

(there is an inner product $\langle\cdot,\cdot\rangle$ on X such that $\|x\| := \sqrt{\langle x, x\rangle}$)

In this case: $\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ for $\mathbb{F} = \mathbb{R}$

$\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\,\|x+iy\|^2 + i\,\|x-iy\|^2\right)$ for $\mathbb{F} = \mathbb{C}$

gives the inner product on X.

Proof: Consider case $\mathbb{F} = \mathbb{R}$. So we define: $\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$.

To show three properties:
(1) positive definite
(2) linear in the second argument
(3) symmetry

(1): $\langle x,x\rangle = \frac{1}{4}\left(\|x+x\|^2 - \|x-x\|^2\right) = \frac{1}{4}\|2\cdot x\|^2 = \|x\|^2 \geq 0$

and $\langle x,x\rangle = 0 \Rightarrow x = 0$

(3): $\langle y,x\rangle = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2\right) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \langle x,y\rangle$

(2) linearity: we will use: $\|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2$

First step: $\langle w,z\rangle = \frac{1}{4}\left(\|w+z\|^2 - \|w-z\|^2\right)$

$= \frac{1}{4}\left(\|w+z\|^2 + \|w\|^2 - \left(\|w\|^2 + \|w-z\|^2\right)\right)$

with $w+z = x+y$, $w = x-y$, $w = \tilde{x}+\tilde{y}$, $w-z = \tilde{x}-\tilde{y}$, where $x := w + \frac{1}{2}z$, $y := \frac{1}{2}z$, $\tilde{x} := w - \frac{1}{2}z$, $\tilde{y} := \frac{1}{2}z$

parallelogram law:

$= \frac{1}{4}\left(2\cdot\|x\|^2 + 2\cdot\|y\|^2 - \left(2\cdot\|\tilde{x}\|^2 + 2\cdot\|\tilde{y}\|^2\right)\right)$

$= \frac{1}{2}\left(\|x\|^2 - \|\tilde{x}\|^2\right) = \frac{1}{2}\left(\|w + \frac{1}{2}z\|^2 - \|w - \frac{1}{2}z\|^2\right)$

$= 2\cdot\langle w, \frac{1}{2}z\rangle$

First result: $\frac{1}{2}\langle w,z\rangle = \langle w, \frac{1}{2}z\rangle$ $\xrightarrow[n\in\mathbb{N}]{\text{induction}}$ $\frac{1}{2^n}\langle w,z\rangle = \langle w, \frac{1}{2^n}z\rangle$ (*)

Additivity: $\langle w,z\rangle + \langle w,\hat{z}\rangle$

$= \frac{1}{4}\left(\|w+z\|^2 - \|w-z\|^2\right) + \frac{1}{4}\left(\|w+\hat{z}\|^2 - \|w-\hat{z}\|^2\right)$

$= \frac{1}{4}\Big(\|w + \frac{z+\hat{z}}{2} + \frac{z-\hat{z}}{2}\|^2 + \|w + \frac{z+\hat{z}}{2} - \frac{z-\hat{z}}{2}\|^2$
$- \left(\|w - \frac{z+\hat{z}}{2} + \frac{z-\hat{z}}{2}\|^2 + \|w - \frac{z+\hat{z}}{2} - \frac{z-\hat{z}}{2}\|^2\right)\Big)$

parallelogram law:

$= \frac{1}{4}\left(2\cdot\|w + \frac{z+\hat{z}}{2}\|^2 + 2\,\|\frac{z-\hat{z}}{2}\|^2 - \left(2\,\|w - \frac{z+\hat{z}}{2}\|^2 + 2\,\|\frac{z-\hat{z}}{2}\|^2\right)\right)$

$= \frac{1}{2}\left(\|w + \frac{z+\hat{z}}{2}\|^2 - \|w - \frac{z+\hat{z}}{2}\|^2\right) = 2\,\langle w, \frac{z+\hat{z}}{2}\rangle$

$\overset{(*)}{=} \langle w, z+\hat{z}\rangle$

Homogeneity: $\langle w,z\rangle + \langle w,z\rangle \overset{\text{additivity}}{=} \langle w, z+z\rangle$

$2\cdot\langle w,z\rangle$ $\qquad$ $\langle w, 2\cdot z\rangle$

$\xrightarrow[k\in\mathbb{N}]{\text{induction}}$ $k\cdot\langle w,z\rangle = \langle w, k\,z\rangle$

combining with (*): $\frac{k}{2^n}\langle w,z\rangle = \langle w, \frac{k}{2^n}z\rangle$ for all $k, n \in \mathbb{N}$

$0\cdot\langle w,z\rangle = \langle w, 0\cdot z\rangle$

$(-1)\cdot\langle w,z\rangle = \langle w, (-1)\cdot z\rangle$

all positive real numbers can be approximated

□