



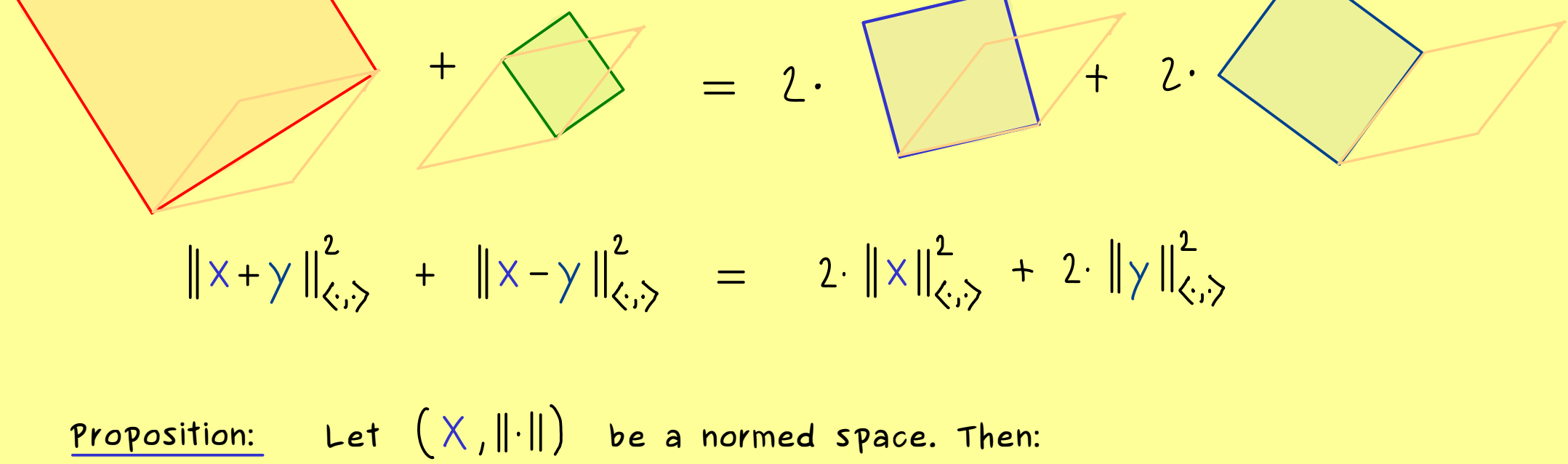
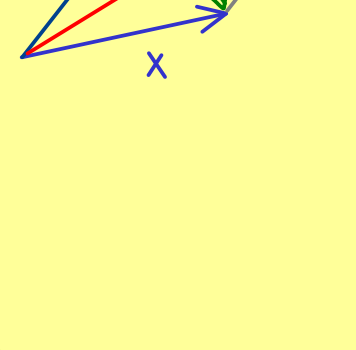
Hilbert Spaces - Part 4

$(X, \langle \cdot, \cdot \rangle)$ inner product space (\mathbb{F} -vector space + inner product)

$$\|x\|_{\langle \cdot, \cdot \rangle} := \sqrt{\langle x, x \rangle} \quad \text{induced norm}$$

We get:

$$\begin{aligned} & \|x+y\|_{\langle \cdot, \cdot \rangle}^2 + \|x-y\|_{\langle \cdot, \cdot \rangle}^2 \\ &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\ &= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle \\ &\quad + \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle \\ &= 2 \cdot \|x\|_{\langle \cdot, \cdot \rangle}^2 + 2 \cdot \|y\|_{\langle \cdot, \cdot \rangle}^2 \quad (\text{parallelogram law}) \end{aligned}$$



$$\|x+y\|_{\langle \cdot, \cdot \rangle}^2 + \|x-y\|_{\langle \cdot, \cdot \rangle}^2 = 2 \cdot \|x\|_{\langle \cdot, \cdot \rangle}^2 + 2 \cdot \|y\|_{\langle \cdot, \cdot \rangle}^2$$

Proposition: Let $(X, \|\cdot\|)$ be a normed space. Then:

the parallelogram law is satisfied $(\forall x, y \in X: \|x+y\|^2 + \|x-y\|^2 = 2 \cdot \|x\|^2 + 2 \cdot \|y\|^2)$

$\iff \|\cdot\|$ is induced by an inner product on X ($\|\cdot\|_{\langle \cdot, \cdot \rangle} = \|\cdot\|$)



In this case: $\langle x, y \rangle := \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2)$ for $\mathbb{F} = \mathbb{R}$

$$\langle x, y \rangle := \frac{1}{4} (\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2)$$

gives the inner product on X . for $\mathbb{F} = \mathbb{C}$

Remember: A Hilbert space is a Banach space where the parallelogram law holds.