# Hilbert Spaces - Part 2

**Definition (Hilbert space):** $(X, \langle \cdot , \cdot \rangle)$ — $\mathbb{F}$ - vector space

$\langle \cdot , \cdot \rangle : X \times X \to \mathbb{F}$ inner product

where $(X, \|\cdot\|)$ is a Banach space

with respect to the norm $\|x\| := \sqrt{\langle x, x \rangle}$

**Example:**

(a) $\mathbb{C}^N$ with standard inner product

(b) $\mathbb{R}^n$ with given inner product

← (finite-dimensional normed vector spaces are always complete)

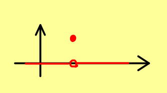
(c) $\ell^2(\mathbb{N}, \mathbb{C}) := \left\{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ ($(x_n)_{n \in \mathbb{N}} = x$)

with inner product: $\langle y, x \rangle = \sum_{n=1}^{\infty} \overline{y_n} \cdot x_n$ (convergent series!)

(d) $(\Omega, \mathcal{A}, \mu)$ measure space

$$\mathcal{L}^2(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \mid \int_\Omega |f|^2 \, d\mu < \infty \right\}$$

$\|f\| := \sqrt{\int_\Omega |f|^2 \, d\mu}$ not a norm in general!



$$L^2(\Omega, \mu) := \mathcal{L}^2(\Omega, \mu) / \mathcal{N} \quad \text{where} \quad \mathcal{N} := \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \mid \|f\| = 0 \right\}$$

$\|[f]\| := \|f\|$ well-defined ⇝ norm on $L^2(\Omega, \mu)$

We get a Hilbert space with the following inner product:

$$\langle [g], [f] \rangle := \int_\Omega \overline{g(\omega)} f(\omega) \, d\mu(\omega)$$