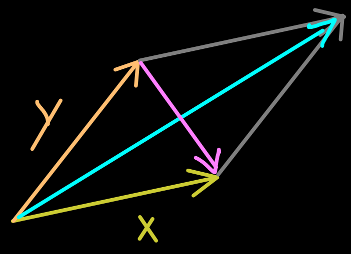
Hilbert Spaces - Part 4

$(X, \langle\cdot,\cdot\rangle)$ inner product space ($\mathbb{F}$ - vector space + inner product)

$$\|x\|_{\langle\cdot,\cdot\rangle} := \sqrt{\langle x, x\rangle} \quad \text{induced norm}$$

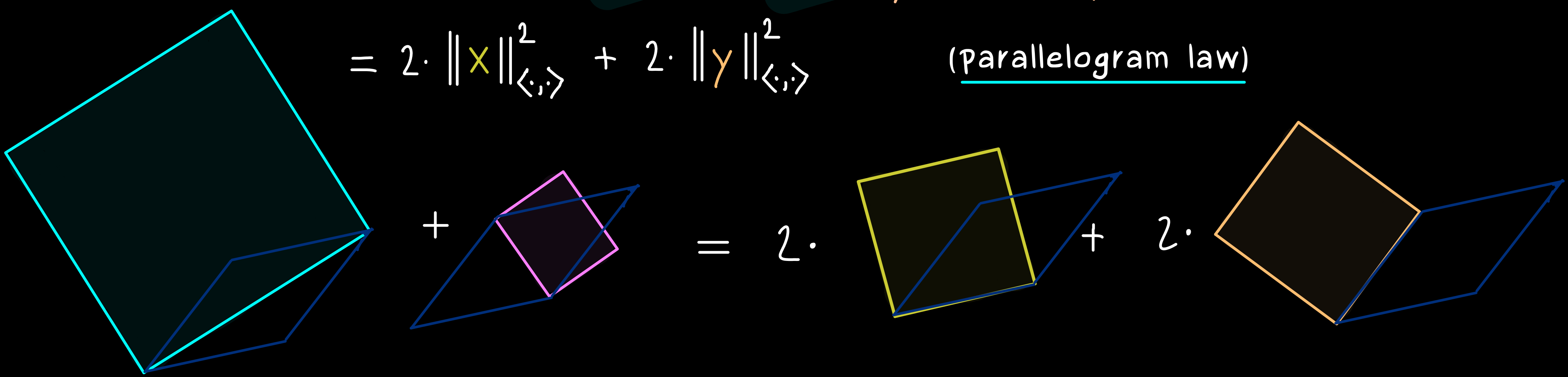
We get: $\|x+y\|^2_{\langle\cdot,\cdot\rangle} + \|x-y\|^2_{\langle\cdot,\cdot\rangle}$



$$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$+ \langle x,x\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle y,y\rangle$$

$$= 2\cdot \|x\|^2_{\langle\cdot,\cdot\rangle} + 2\cdot\|y\|^2_{\langle\cdot,\cdot\rangle} \quad \text{(parallelogram law)}$$



$$\|x+y\|^2_{\langle\cdot,\cdot\rangle} + \|x-y\|^2_{\langle\cdot,\cdot\rangle} = 2\cdot\|x\|^2_{\langle\cdot,\cdot\rangle} + 2\cdot\|y\|^2_{\langle\cdot,\cdot\rangle}$$

Proposition: Let $(X, \|\cdot\|)$ be a normed space. Then:

the parallelogram law is satisfied $\left(\forall x,y \in X: \|x+y\|^2 + \|x-y\|^2 = 2\cdot\|x\|^2 + 2\cdot\|y\|^2\right)$

$\Longleftrightarrow$ $\|\cdot\|$ is induced by an inner product on X $\quad \left(\|\cdot\|_{\langle\cdot,\cdot\rangle} = \|\cdot\|\right)$

next video

In this case: $\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$ for $\mathbb{F} = \mathbb{R}$

$\langle x,y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\,\|x+iy\|^2 + i\,\|x-iy\|^2\right)$ for $\mathbb{F} = \mathbb{C}$

gives the inner product on X.

Remember: A Hilbert space is a Banach space where the parallelogram law holds.