Hilbert Spaces - Part 3

$(X, \langle \cdot,\cdot \rangle)$ inner product space ($\mathbb{F}$ - vector space + inner product)

$\Longrightarrow$ $(X, \|\cdot\|)$ normed space with $\|x\| := \sqrt{\langle x, x\rangle}$

norm induced by inner product

Polarization identity: (for case $\mathbb{F} = \mathbb{C}$)

$(X, \langle \cdot,\cdot \rangle)$ inner product space with induced norm $\|\cdot\|$. Then, for all $x, y \in X$:

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\,\|x+iy\|^2 + i\,\|x-iy\|^2\right)$$

inner product is linear in the second argument

Proof:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$

$$-\|x-y\|^2 = -\langle x-y, x-y\rangle = -\langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle - \langle y,y\rangle$$

$$-i\cdot\|x+iy\|^2 = -i\langle x+iy, x+iy\rangle = -i\langle x,x\rangle - \langle y,x\rangle + \langle x,y\rangle - i\langle y,y\rangle$$

$$i\,\|x-iy\|^2 = i\langle x-iy, x-iy\rangle = i\langle x,x\rangle - \langle y,x\rangle + \langle x,y\rangle + i\langle y,y\rangle$$

$\square$

Polarization identity: (for case $\mathbb{F} = \mathbb{R}$)

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \quad \text{for all } x, y \in X.$$